كلية المأمون الجامعة
قسم هندسة تقنيات الحاسوب

CONTROL SYSTEM

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Lesson 2

Basics of Laplace Transform

Basics of Laplace Transform

Objectives:

At the end of this lesson, students will be able to:

1. Use Laplace Transform in solving differential equations of Control System.

Basics of Laplace Transform

The transformation technique relating the time functions to frequency dependent functions of a complex variable is called the **Laplace transformation technique.** Such transformation is very useful in solving linear differential equations.

Definition of Laplace Transform

The Laplace transform is defined as below :

Let f(t) be a real function of a real variable t defined for t > 0, then

$$F(s) = L\{f(t)\} = \int_0^\infty f(t).e^{-st}\,dt$$

where F(s) is called Laplace transform of f(t). And the variable 's' which appears in F(s) is frequency dependent complex variable.

The time function f(t) is obtained back from the Laplace transform by a process called Inverse Laplace transform and denoted as L^{-1}.

$$L^{-1}[F(s)] = L^{-1}\{L(f(t))\} = f(t)$$

Example 2.1 : *Find the Laplace transform of e^{-at} and 1 for $t \geq 0$.*

Solution : i) $f(t) = e^{-at}$

$$F(s) = L\{f(t)\} = \int_0^\infty f(t)\,e^{-st}\,dt = \int_0^\infty e^{-at}\cdot e^{-st}\,dt$$

$$= \int_0^\infty e^{-(s+a)t}\,dt = \left[-\frac{1}{(s+a)}\cdot e^{-(s+a)t}\right]_0^\infty$$

$$= 0 - \left(\frac{-1}{s+a}\right) = \frac{1}{s+a}$$

$$\therefore \quad L\{e^{-at}\} = \frac{1}{s+a} \quad \text{and} \quad L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

(ii) $f(t) = 1$

$$\therefore \quad F(s) = \int_0^\infty f(t)\,e^{-st}\,dt = \int_0^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}$$

$$\therefore \quad L\{1\} = \frac{1}{s} \quad \text{and} \quad L^{-1}\left\{\frac{1}{s}\right\} = 1$$

Properties of Laplace Transform

1. Linearity

So if $F_1(s)$, $F_2(s)$, ..., $F_n(s)$ are the Laplace transforms of the time functions $f_1(t)$, $f_2(t)$, ..., $f_n(t)$ respectively then,

$$L\{f_1(t) + f_2(t) + \dots + f_n(t)\} = F_1(s) + F_2(s) + \dots + F_n(s)$$

2. Scaling Theorem

$$L\{K\, f(t)\} = K\, F(s)$$ ... K is constant

3. Real Differentiation

Let F(s) be the Laplace transform of f(t). Then,

$$L\left\{\frac{d\, f(t)}{dt}\right\} = s\, F(s) - f(0^-)$$

where $f(0^-)$ indicates value of f(t) at $t = 0^-$ i.e. just before the instant $t = 0$

The theorem can be extended for n^{th} order derivative as,

$$L\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \dots - f^{(n-1)}(0^-)$$

where $f^{(n-1)}(0^-)$ is the value of $(n-1)^{th}$ derivative of f(t) at $t = 0^-$.

i.e for n = 2, $L\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2 F(s) - s\, f(0^-) - f'(0^-)$

for n = 3, $L\left\{\frac{d^3 f(t)}{dt^3}\right\} = s^3 F(s) - s^2 f(0^-) - s f'(0^-) - f''(0^-)$ and so on.

4. Real Integration

If F(s) is the Laplace transform of f(t) then,

$$L\left\{\int_0^t f(t)\, dt\right\} = \frac{F(s)}{s}$$

5. Differentiation by S

$$L\{t\,f(t)\} = \frac{-dF(s)}{ds}$$

Thus,

$$L\{t\} = L\{t \times 1\} = -\frac{d}{ds}[L\{1\}] = -\frac{d}{ds}\left[\frac{1}{s}\right] = \frac{1}{s^2} = \frac{1!}{s^{1+1}}$$

$$L\{t^2\} = L\{t \times t\} = -\frac{d}{ds}[L\{t\}] = -\frac{d}{ds}\left[\frac{1}{s^2}\right] = \frac{2}{s^3} = \frac{2!}{s^{2+1}}$$

6. Complex Translation

$$F(s-a) = L\{e^{at} f(t)\}$$ and $$F(s+a) = L\{e^{-at} f(t)\}$$

$$F(s \mp a) = F(s)\big|_{s = s \mp a}$$

where F(s) is the Laplace transform of f(t).

7. Real Translation (Shifting Theorem)

This theorem is useful to obtain the Laplace transform of the shifted or delayed function of time.

If F(s) is the Laplace transform of f(t) then the Laplace transform of the function **delayed by time T** is,

$$L\{f(t-T)\} = e^{-Ts} F(s)$$

8. Initial Value Theorem

The Laplace transform is very useful to find the initial value of the time function f(t). Thus if F(s) is the Laplace transform of f(t) then,

$$f(0^+) = \lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

9. Final Value Theorem

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Table of Laplace Transforms :

f(t)	F(s)
1	$\frac{1}{s}$
Constant K	$\frac{K}{s}$
K f(t), K is constant	K F(s)
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
e^{at}	$\frac{1}{s-a}$
$e^{-at} t^n$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t$	$\frac{(s+a)}{(s+a)^2+\omega^2}$
$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
$t\, e^{-at}$	$\frac{1}{(s+a)^2}$
$1 - e^{-at}$	$\frac{a}{s(s+a)}$

Function f(t)	Laplace Transform F(s)
Unit step = u(t)	$\frac{1}{s}$
A u(t)	$\frac{A}{s}$
Delayed unit step = u(t − T)	$\frac{e^{-Ts}}{s}$
A u(t − T)	$\frac{A e^{-Ts}}{s}$
Unit ramp = r(t) = t u(t)	$\frac{1}{s^2}$
A t u(t)	$\frac{A}{s^2}$
Delayed unit ramp = r(t − T) = (t − T) u(t − T)	$\frac{e^{-Ts}}{s^2}$
A (t − T) u(t − T)	$\frac{A e^{-Ts}}{s^2}$
Unit impulse = δ(t)	1
Delayed unit impulse = δ(t−T)	e^{-Ts}
Impulse of strength K i.e. K δ(t)	K

Inverse Laplace Transform

As mentioned earlier, inverse Laplace transform is calculated by partial fraction method rather than complex integration evaluation. Let F(s) is the Laplace transform of f(t) then the inverse Laplace transform is denoted as,

$$f(t) = L^{-1}[F(s)]$$

The F(s), in partial fraction method, is written in the form as,

$$F(s) = \frac{N(s)}{D(s)}$$

where N(s) = Numerator polynomial in s

and D(s) = Denominator polynomial in s

The given function F(s) can be expressed in partial fraction form only when **degree of N(s) is less than D(s).** Hence if degree of N(s) is equal or higher than D(s) then mathematically divide N(s) by D(s) to express F(s) in quotient and remainder form as,

$$F(s) = Q + F_1(s) = Q + \frac{N'(s)}{D'(s)}$$

where Q = Quotient obtained by dividing N(s) by D(s)

and $F_1(s) = \frac{N'(s)}{D'(s)}$ = Remainder

There are 3 types of D(s):

1. Simple and Real Roots

The roots of D(s) are simple and real. Hence the function F(s) can be expressed as,

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s-a)(s-b)(s-c)\ldots}$$

Example 2.2 : *Find the inverse Laplace transform of given F(s).*

$$F(s) = \frac{(s+2)}{s(s+3)(s+4)}$$

Solution : The degree of N(s) is less than D(s). Hence F(s) can be expressed as,

$$F(s) = \frac{K_1}{s} + \frac{K_2}{(s+3)} + \frac{K_3}{(s+4)}$$

where $K_1 = s \cdot F(s)\big|_{s=0} = s \cdot \frac{(s+2)}{s(s+3)(s+4)}\bigg|_{s=0} = \frac{2}{3\times 4} = \frac{1}{6}$

$$K_2 = (s+3) \cdot F(s)\big|_{s=-3} = (s+3) \cdot \frac{(s+2)}{s(s+3)(s+4)}\bigg|_{s=-3} = \frac{(-3+2)}{(-3)(-3+4)} = \frac{1}{3}$$

$$K_3 = (s+4) \cdot F(s)\big|_{s=-4} = (s+4) \cdot \frac{(s+2)}{s(s+3)(s+4)}\bigg|_{s=-4} = \frac{(-4+2)}{(-4)(-4+3)} = -\frac{1}{2}$$

$$\therefore \quad F(s) = \frac{1/6}{s} + \frac{1/3}{(s+3)} - \frac{1/2}{(s+4)}$$

Taking inverse Laplace transform,

$$\therefore \quad f(t) = \frac{1}{6} + \frac{1}{3}e^{-3t} - \frac{1}{2}e^{-4t}$$

2. Multiple Roots

The given function is of the form,

$$F(s) = \frac{N(s)}{(s-a)^n \ D'(s)}$$

Example 2.3 : *Obtain the inverse Laplace transform of given F(s).*

$$F(s) = \frac{(s-2)}{s(s+1)^3}$$

Solution : The given F(s) can be expressed as,

$$F(s) = \frac{K_0}{(s+1)^3} + \frac{K_1}{(s+1)^2} + \frac{K_2}{(s+1)} + \frac{K_3}{s}$$

Finding L.C.M. of right hand side,

$$\frac{(s-2)}{s(s+1)^3} = \frac{K_0(s) + K_1(s+1)s + K_2(s+1)^2 s + K_3(s+1)^3}{s(s+1)^3}$$

$$\therefore (s-2) = K_0 s + K_1 s^2 + K_1 s + K_2 s^3 + 2K_2 s^2 + K_2 s + K_3 s^3 + 3K_3 s^2 + 3K_3 s + K_3$$

Comparing coefficients of various powers of s on both sides,

for s^3, $K_2 + K_3 = 0$ ... (1)

for s^2, $K_1 + 2K_2 + 3K_3 = 0$ ... (2)

for s^1, $K_0 + K_1 + K_2 + 3K_3 = 1$ ... (3)

for s^0, $K_3 = -2$ ... (4)

As $K_3 = -2$

from (1), $K_2 = 2$

$\therefore$ from (2), $K_1 = 2$

$\therefore$ from (3), $K_0 = 3$

$$\therefore \quad F(s) = \frac{3}{(s+1)^3} + \frac{2}{(s+1)^2} + \frac{2}{(s+1)} - \frac{2}{s}$$

Now $$L[e^{-at} t^n] = \frac{n!}{(s+a)^{n+1}}$$

$$\therefore \quad L^{-1}\left[\frac{1}{(s+a)^{n+1}}\right] = \frac{e^{-at} t^n}{n!}$$

$$\therefore \quad F(s) = 3 \cdot \frac{1}{(s+1)^3} + 2 \cdot \frac{1}{(s+1)^2} + 2 \cdot \frac{1}{(s+1)} - 2 \cdot \frac{1}{s}$$

$$\therefore \quad f(t) = L^{-1}[F(s)] = \frac{3}{2!} e^{-t} \cdot t^2 + \frac{2}{1!} e^{-t} \cdot t + 2e^{-t} - 2$$

$$\therefore \quad \mathbf{f(t) = \frac{3}{2} t^2 e^{-t} + 2 t e^{-t} + 2 e^{-t} - 2}$$

3. Complex Conjugate Roots

If there exists a quadratic term in D(s) of F(s) whose roots are complex conjugates then the F(s) is expressed with a first order polynomial in s in the numerator as,

$$F(s) = \frac{As+B}{(s^2+\alpha s+\beta)} + \frac{N'(s)}{D'(s)}$$

where $(s^2+\alpha s+\beta)$ is the quadratic whose **roots are complex conjugates** while $\frac{N'(s)}{D'(s)}$ represents remaining terms of the expansion. The A and B are partial fraction coefficients.

Consider $\quad F_1(s) = \frac{As+B}{s^2+\alpha s+\beta} \quad$ A and B are known

Now complete the square in the denominator by calculating last term as,

$$L.T. = \frac{(M.T.)^2}{4\,(F.T.)}$$

where L.T = Last term

M.T = Middle term

F.T = First term

$$\therefore \quad L.T. = \frac{\alpha^2}{4}$$

$$\therefore \quad F_1(s) = \frac{As+B}{s^2+\alpha s+\frac{\alpha^2}{4}+\beta-\frac{\alpha^2}{4}} = \frac{As+B}{\left(s+\frac{\alpha}{2}\right)^2+\omega^2}$$

where $\quad \omega = \sqrt{\beta - \frac{\alpha^2}{4}}$

Now adjust the numerator As + B in such a way that it is of the form,

$$L\left[e^{-at}\sin\omega t\right] = \frac{\omega}{(s+a)^2+\omega^2} \quad \text{or} \quad L\left[e^{-at}\cos\omega t\right] = \frac{(s+a)}{(s+a)^2+\omega^2}$$

Key Point: *Thus inverse Laplace transform of F(s) having complex conjugate roots of D(s), always contains sine, cosine or damped sine or damped cosine functions.*

Example 2.4 : *Find the inverse Laplace transform of*

$$F(s) = \frac{s^2+3}{(s^2+2s+5)(s+2)}$$

Solution : The given F(s) can be written as,

$$F(s) = \frac{As+B}{s^2+2s+5} + \frac{C}{s+2}$$

as s^2+2s+5 has complex conjugate roots. To find A, B and C find L.C.M. of right hand side,

$$\therefore F(s) = \frac{(s+2)(As+B)+C(s^2+2s+5)}{(s^2+2s+5)(s+2)}$$

$$\therefore \frac{s^2+3}{(s^2+2s+5)(s+2)} = \frac{As^2+2As+Bs+2B+Cs^2+2sC+5C}{(s^2+2s+5)(s+2)}$$

Comparing the coefficients of various powers of s, of the numerators of both sides

$$\therefore s^2+3 = s^2(A+C)+s(2A+B+2C)+(2B+5C)$$

$$\therefore A + C = 1 \quad \ldots(1)$$

$$\therefore 2A + B + 2C = 0 \quad \ldots(2)$$

$$\therefore 2B + 5C = 3 \quad \ldots(3)$$

To solve the equations quickly, the coefficient C corresponding to the simple, real root can be obtained as,

$$\therefore C = F(s).(s+2)\Big|_{s=-2} = \left.\frac{(s^2+3)(s+2)}{(s^2+2s+5)(s+2)}\right|_{s=-2} = \frac{(4+3)}{(4-4+5)} = \frac{7}{5}$$

Substituting in (1) and (2),

$$A = -\frac{2}{5}$$

and

$$B = -2$$

$$\therefore F(s) = \frac{-\frac{2}{5}s-2}{s^2+2s+5} + \frac{\frac{7}{5}}{(s+2)}$$

Consider $$F_1(s) = \frac{-\frac{2}{5}s-2}{s^2+2s+5}$$

Completing square in the denominator,

$$F_1(s) = \frac{-\frac{2}{5}s-2}{s^2+2s+1+5-1} = \frac{-\frac{2}{5}s-2}{(s+1)^2+(2)^2} = -\frac{2}{5}\left[\frac{s+5}{(s+1)^2+(2)^2}\right]$$

$$= -\frac{2}{5}\left[\frac{s+1+4}{(s+1)^2+(2)^2}\right] \qquad \text{split 4 as } 2\times 2$$

$$= -\frac{2}{5}\left[\frac{s+1}{(s+1)^2+(2)^2}+2\times\frac{2}{(s+1)^2+(2)^2}\right]$$

$$\therefore\ F(s) = -\frac{2}{5}\left\{\frac{(s+1)}{(s+1)^2+(2)^2}+2\cdot\frac{2}{(s+1)^2+(2)^2}\right\}+\frac{\frac{7}{5}}{(s+2)}$$

As $$L^{-1}\left[\frac{(s+a)}{(s+a)^2+\omega^2}\right] = \left[e^{-at}\cos\omega t\right] \quad \text{and}$$

$$L^{-1}\left[\frac{\omega}{(s+a)^2+\omega^2}\right] = \left[e^{-at}\sin\omega t\right]$$

Hence taking inverse Laplace transform of F(s),

$$f(t) = -\frac{2}{5}\left[e^{-t}\cos 2t+2\,e^{-t}\sin 2t\right]+\frac{7}{5}e^{-2t}$$

Use of Laplace Transform in Control System

Example 2.5 : *Obtain the expression for y(t) which is satisfying the differential equation* $\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 16e^{-t}$. *Neglect initial conditions.*

Solution : Taking Laplace transform of both sides of the given differential equation and neglecting initial condition terms in Laplace transform of $\frac{d^2y(t)}{dt^2}$ and $\frac{dy(t)}{dt}$ we get,

$$s^2Y(s) + 6sY(s) + 8Y(s) = \frac{16}{s+1}$$

$$\therefore \quad (s^2+6s+8)\,Y(s) = \frac{16}{(s+1)}$$

$$\therefore \quad Y(s) = \frac{16}{(s+1)(s^2+6s+8)}$$

$$\therefore \quad Y(s) = \frac{16}{(s+1)(s+2)(s+4)}$$

$$\therefore \quad Y(s) = \frac{a_1}{s+1} + \frac{a_2}{s+2} + \frac{a_3}{s+4}$$

$$\therefore \quad Y(s) = \frac{5.33}{s+1} - \frac{8}{s+2} + \frac{2.66}{s+4}$$

Taking inverse Laplace transform of Y(s),

$$\mathbf{y(t) = 5.33\,e^{-t} - 8\,e^{-2t} + 2.66\,e^{-4t}}$$

This is the required solution of differential equation.

Special Case of Inverse Laplace Transform

$$F(s) = \frac{P(s)}{Q(s)} \qquad \text{... order of P(s) and Q(s) same}$$

$$= K + \frac{P'(s)}{Q(s)} \qquad \text{.... after dividing P(s) by Q(s)}$$

Now Laplace inverse of constant term is impulse function. Refer last pair in the Table 2.2

$$\therefore \quad L^{-1}\{K\} = K\delta(t) \qquad \text{where } \delta(t) = \text{unit impulse.}$$

While P' (s) / Q(s) can now be expressed to obtain partial fraction expansion, to get its inverse very easily.

Example 2.6 : *Find the Laplace inverse of* $F(s) = \dfrac{s^3+18s^2+3s+5}{s^3+8s^2+17s+10}$.

Solution : Divide P(s) by Q(s).

$$
\begin{array}{r|l}
s^3+8s^2+17s+10\,) & s^3+18s^2+3s+5\;(1 \rightarrow K \\
 & s^3+8s^2+17s+10 \\
 & -\quad -\quad -\quad - \\
\hline
 & 10s^2-14s-5 \;\rightarrow P'(s)
\end{array}
$$

$$\therefore \quad F(s) = 1+\frac{10s^2-14s-5}{s^3+8s^2+17s+10}$$

$$= 1+\frac{10s^2-14s-5}{(s+2)\,(s+1)\,(s+5)} = 1+\frac{A}{s+2}+\frac{B}{s+1}+\frac{C}{s+5}$$

$$\therefore \quad A = \left.\frac{10s^2-14s-5}{(s+1)\,(s+5)}\right|_{s=-2} = -21$$

$$B = \left.\frac{10s^2-14s-5}{(s+2)\,(s+5)}\right|_{s=-1} = 4.75$$

$$C = \left.\frac{10s^2-14s-5}{(s+1)\,(s+2)}\right|_{s=-5} = 26.25$$

$$\therefore \quad F(s) = 1-\frac{21}{s+2}+\frac{4.75}{s+1}+\frac{26.25}{s+5}$$

$$\therefore \quad f(t) = L^{-1}\{F(s)\} = \delta(t)-21\,e^{-2t}+4.75\,e^{-t}+26.25\,e^{-5t}$$

where $L^{-1}\{1\} = \delta(t)$ = Unit impulse function.

Example 2.7 : *Find the inverse Laplace transform of,*

$$F(s) = \frac{2s+5}{s^2+5s+6}$$

(PTU, Jan.-2006)

Solution : Factorising denominator,

$$F(s) = \frac{2s+5}{(s+2)(s+3)} = \frac{A}{s+2}+\frac{B}{s+3} \qquad \text{... Partial fractions}$$

$$A = F(s)\,(s+2)\big|_{s=-2} = \frac{(-4+5)}{(3-2)} = 1$$

$$B = F(s)\,(s+3)\big|_{s=-3} = \frac{(-6+5)}{(-3+2)} = 1$$

$$\therefore \quad F(s) = \frac{1}{s+2}+\frac{1}{s+3}$$

$$\therefore \quad f(t) = L^{-1}\{F(s)\} = e^{-2t}+e^{-3t}$$

Lesson 3

Mathematical Modelling of Control System