6

Parallel Circuits

6.1 INTRODUCTION

Two network configurations, series and parallel, form the framework for some of the most complex network structures. A clear understanding of each will pay enormous dividends as more complex methods and networks are examined. The series connection was discussed in detail in the last chapter. We will now examine the **parallel circuit** and all the methods and laws associated with this important configuration.

6.2 PARALLEL ELEMENTS

Two elements, branches, or networks are in* parallel *if they have two points in common.

In Fig. 6.1, for example, elements 1 and 2 have terminals *a* and *b* in common; they are therefore in parallel.

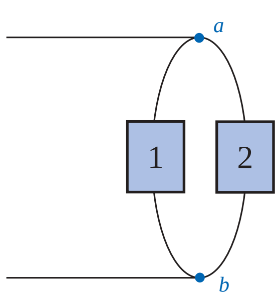

FIG. 6.1
Parallel elements.

In Fig. 6.2, all the elements are in parallel because they satisfy the above criterion. Three configurations are provided to demonstrate how the parallel networks can be drawn. Do not let the squaring of the con-

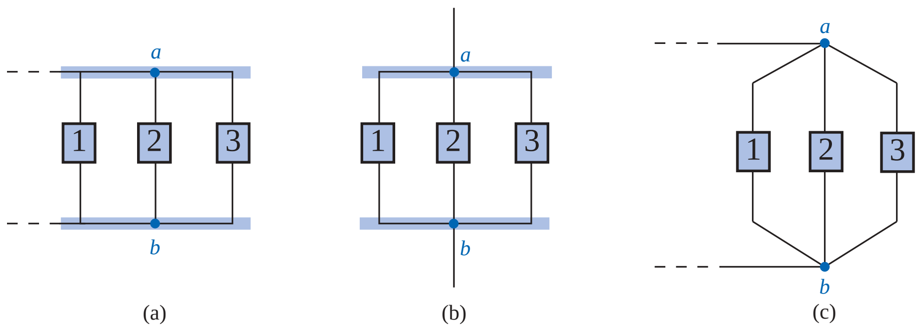

FIG. 6.2
Different ways in which three parallel elements may appear.

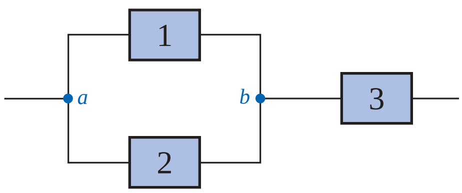

FIG. 6.3
Network in which 1 and 2 are in parallel and 3 is in series with the parallel combination of 1 and 2.

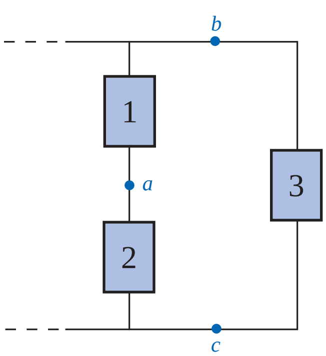

FIG. 6.4
Network in which 1 and 2 are in series and 3 is in parallel with the series combination of 1 and 2.

nection at the top and bottom of Fig. 6.2(a) and (b) cloud the fact that all the elements are connected to one terminal point at the top and bottom, as shown in Fig. 6.2(c).

In Fig. 6.3, elements 1 and 2 are in parallel because they have terminals *a* and *b* in common. The parallel combination of 1 and 2 is then in series with element 3 due to the common terminal point *b*.

In Fig. 6.4, elements 1 and 2 are in series due to the common point *a*, but the series combination of 1 and 2 is in parallel with element 3 as defined by the common terminal connections at *b* and *c*.

In Figs. 6.1 through 6.4, the numbered boxes were used as a general symbol representing single resistive elements, or batteries, or complex network configurations.

Common examples of parallel elements include the rungs of a ladder, the tying of more than one rope between two points to increase the strength of the connection, and the use of pipes between two points to split the water between the two points at a ratio determined by the area of the pipes.

6.3 TOTAL CONDUCTANCE AND RESISTANCE

Recall that for series resistors, the total resistance is the sum of the resistor values.

For parallel elements, the total conductance is the sum of the individual conductances.

That is, for the parallel network of Fig. 6.5, we write

$$G_T = G_1 + G_2 + G_3 + \cdots + G_N \tag{6.1}$$

Since increasing levels of conductance will establish higher current levels, the more terms appearing in Eq. (6.1), the higher the input cur-

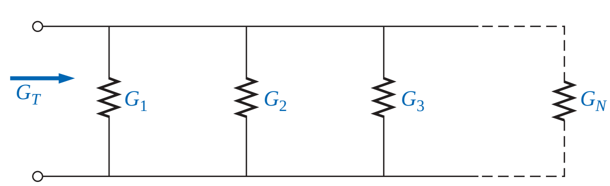

FIG. 6.5
Determining the total conductance of parallel conductances.

rent level. In other words, as the number of resistors in parallel increases, the input current level will increase for the same applied voltage—the opposite effect of increasing the number of resistors in series.

Substituting resistor values for the network of Fig. 6.5 will result in the network of Fig. 6.6. Since $G = 1/R$, the total resistance for the network can be determined by direct substitution into Eq. (6.1):

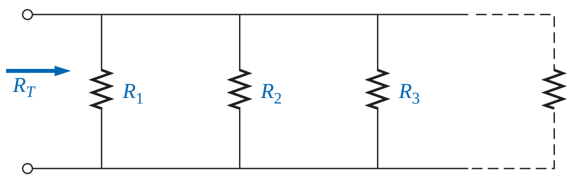

FIG. 6.6
Determining the total resistance of parallel resistors.

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots + \frac{1}{R_N} \qquad \textbf{(6.2)}$$

Note that the equation is for 1 divided by the total resistance rather than the total resistance. Once the sum of the terms to the right of the equals sign has been determined, it will then be necessary to divide the result into 1 to determine the total resistance. The following examples will demonstrate the additional calculations introduced by the inverse relationship.

EXAMPLE 6.1 Determine the total conductance and resistance for the parallel network of Fig. 6.7.

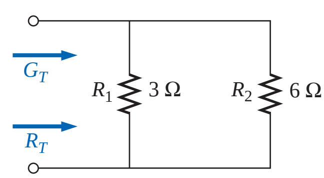

FIG. 6.7
Example 6.1.

Solution:

$$G_T = G_1 + G_2 = \frac{1}{3\,\Omega} + \frac{1}{6\,\Omega} = 0.333\text{ S} + 0.167\text{ S} = \mathbf{0.5\text{ S}}$$

and
$$R_T = \frac{1}{G_T} = \frac{1}{0.5\text{ S}} = \mathbf{2\,\Omega}$$

EXAMPLE 6.2 Determine the effect on the total conductance and resistance of the network of Fig. 6.7 if another resistor of 10 Ω were added in parallel with the other elements.

Solution:

$$G_T = 0.5\text{ S} + \frac{1}{10\,\Omega} = 0.5\text{ S} + 0.1\text{ S} = \mathbf{0.6\text{ S}}$$

$$R_T = \frac{1}{G_T} = \frac{1}{0.6\text{ S}} \cong \mathbf{1.667\,\Omega}$$

Note, as mentioned above, that adding additional terms increases the conductance level and decreases the resistance level.

EXAMPLE 6.3 Determine the total resistance for the network of Fig. 6.8.

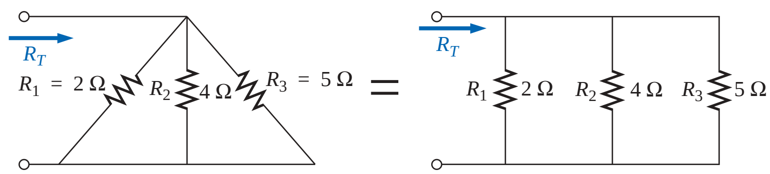

FIG. 6.8
Example 6.3.

Solution:

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

$$= \frac{1}{2\ \Omega} + \frac{1}{4\ \Omega} + \frac{1}{5\ \Omega} = 0.5\ \text{S} + 0.25\ \text{S} + 0.2\ \text{S}$$

$$= 0.95\ \text{S}$$

and $$R_T = \frac{1}{0.95\ \text{S}} = \mathbf{1.053\ \Omega}$$

The above examples demonstrate an interesting and useful (for checking purposes) characteristic of parallel resistors:

The total resistance of parallel resistors is always less than the value of the smallest resistor.

In addition, the wider the spread in numerical value between two parallel resistors, the closer the total resistance will be to the smaller resistor. For instance, the total resistance of 3 Ω in parallel with 6 Ω is 2 Ω, as demonstrated in Example 6.1. However, the total resistance of 3 Ω in parallel with 60 Ω is 2.85 Ω, which is much closer to the value of the smaller resistor.

For *equal* resistors in parallel, the equation becomes significantly easier to apply. For N equal resistors in parallel, Equation (6.2) becomes

$$\frac{1}{R_T} = \underbrace{\frac{1}{R} + \frac{1}{R} + \frac{1}{R} + \cdots + \frac{1}{R}}_{N}$$

$$= N\left(\frac{1}{R}\right)$$

and $$R_T = \frac{R}{N} \tag{6.3}$$

In other words, the total resistance of N parallel resistors of equal value is the resistance of *one* resistor divided by the number (N) of parallel elements.

For conductance levels, we have

$$G_T = NG \tag{6.4}$$

EXAMPLE 6.4

a. Find the total resistance of the network of Fig. 6.9.
b. Calculate the total resistance for the network of Fig. 6.10.

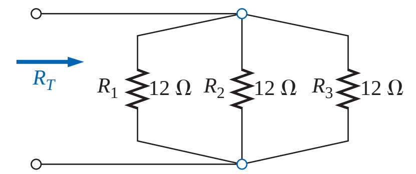

FIG. 6.9
Example 6.4: three parallel resistors of equal value.

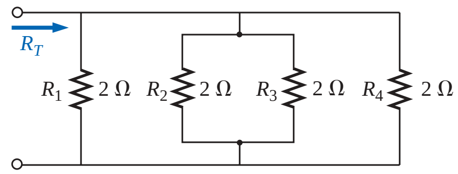

FIG. 6.10
Example 6.4: four parallel resistors of equal value.

Solutions:

a. Figure 6.9 is redrawn in Fig. 6.11:

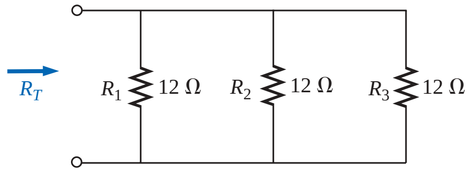

FIG. 6.11
Redrawing the network of Fig. 6.9.

$$R_T = \frac{R}{N} = \frac{12\ \Omega}{3} = \mathbf{4\ \Omega}$$

b. Figure 6.10 is redrawn in Fig. 6.12:

$$R_T = \frac{R}{N} = \frac{2\ \Omega}{4} = \mathbf{0.5\ \Omega}$$

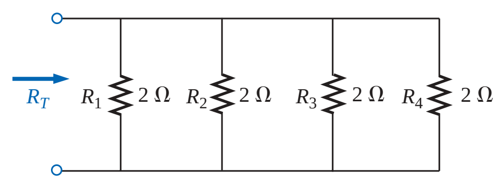

FIG. 6.12
Redrawing the network of Fig. 6.10.

In the vast majority of situations, only two or three parallel resistive elements need to be combined. With this in mind, the following equations were developed to reduce the effects of the inverse relationship when determining R_T.

For two parallel resistors, we write

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2}$$

Multiplying the top and bottom of each term of the right side of the equation by the other resistor will result in

$$\frac{1}{R_T} = \left(\frac{R_2}{R_2}\right)\frac{1}{R_1} + \left(\frac{R_1}{R_1}\right)\frac{1}{R_2} = \frac{R_2}{R_1R_2} + \frac{R_1}{R_1R_2}$$
$$= \frac{R_2 + R_1}{R_1R_2}$$

and

$$R_T = \frac{R_1R_2}{R_1 + R_2} \qquad \textbf{(6.5)}$$

In words,

the total resistance of two parallel resistors is the product of the two divided by their sum.

For three parallel resistors, the equation for R_T becomes

$$R_T = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}} \qquad \textbf{(6.6a)}$$

requiring that we be careful with all the divisions into 1.

Equation (6.6a) can also be expanded into the form of Eq. (6.5), resulting in Eq. (6.6b):

$$R_T = \frac{R_1 R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} \quad \textbf{(6.6b)}$$

with the denominator showing all the possible product combinations of the resistors taken two at a time. An alternative to either form of Eq. (6.6) is to simply apply Eq. (6.5) twice, as will be demonstrated in Example 6.6.

EXAMPLE 6.5 Repeat Example 6.1 using Eq. (6.5).

Solution:

$$R_T = \frac{R_1 R_2}{R_1 + R_2} = \frac{(3\ \Omega)(6\ \Omega)}{3\ \Omega + 6\ \Omega} = \frac{18\ \Omega}{9} = \mathbf{2\ \Omega}$$

EXAMPLE 6.6 Repeat Example 6.3 using Eq. (6.6a).

Solution:

$$\begin{aligned} R_T &= \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}} \\ &= \frac{1}{\dfrac{1}{2\ \Omega} + \dfrac{1}{4\ \Omega} + \dfrac{1}{5\ \Omega}} = \frac{1}{0.5 + 0.25 + 0.2} \\ &= \frac{1}{0.95} = \mathbf{1.053\ \Omega} \end{aligned}$$

Applying Eq. (6.5) twice yields

$$R'_T = 2\ \Omega \parallel 4\ \Omega = \frac{(2\ \Omega)(4\ \Omega)}{2\ \Omega + 4\ \Omega} = \frac{4}{3}\ \Omega$$

$$R_T = R'_T \parallel 5\ \Omega = \frac{\left(\dfrac{4}{3}\ \Omega\right)\left(5\ \Omega\right)}{\dfrac{4}{3}\ \Omega + 5\ \Omega} = \mathbf{1.053\ \Omega}$$

Recall that series elements can be interchanged without affecting the magnitude of the total resistance or current. In parallel networks,

parallel elements can be interchanged without changing the total resistance or input current.

Note in the next example how redrawing the network can often clarify which operations and equations should be applied.

EXAMPLE 6.7 Calculate the total resistance of the parallel network of Fig. 6.13.

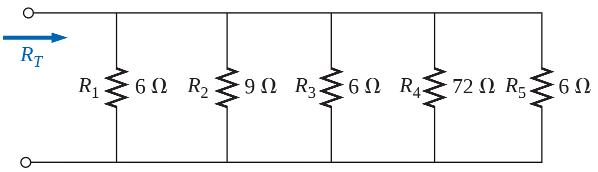

FIG. 6.13
Example 6.7.

Solution: The network is redrawn in Fig. 6.14:

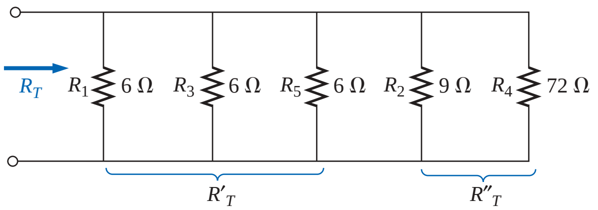

FIG. 6.14
Network of Fig. 6.13 redrawn.

$$R'_T = \frac{R}{N} = \frac{6\ \Omega}{3} = 2\ \Omega$$

$$R''_T = \frac{R_2 R_4}{R_2 + R_4} = \frac{(9\ \Omega)(72\ \Omega)}{9\ \Omega + 72\ \Omega} = \frac{648\ \Omega}{81} = 8\ \Omega$$

and $R_T = R'_T \parallel R''_T$ (In parallel with)

$$= \frac{R'_T R''_T}{R'_T + R''_T} = \frac{(2\ \Omega)(8\ \Omega)}{2\ \Omega + 8\ \Omega} = \frac{16\ \Omega}{10} = \mathbf{1.6\ \Omega}$$

The preceding examples show direct substitution, in which once the proper equation is defined, it is only a matter of plugging in the numbers and performing the required algebraic maneuvers. The next two examples have a design orientation, where specific network parameters are defined and the circuit elements must be determined.

EXAMPLE 6.8 Determine the value of R_2 in Fig. 6.15 to establish a total resistance of 9 kΩ.

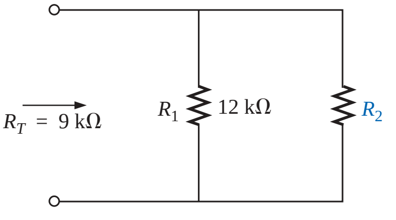

FIG. 6.15
Example 6.8.

Solution:

$$R_T = \frac{R_1 R_2}{R_1 + R_2}$$

$$R_T(R_1 + R_2) = R_1 R_2$$

$$R_T R_1 + R_T R_2 = R_1 R_2$$

$$R_T R_1 = R_1 R_2 - R_T R_2$$

$$R_T R_1 = (R_1 - R_T) R_2$$

and

$$R_2 = \frac{R_T R_1}{R_1 - R_T} \qquad \textbf{(6.7)}$$

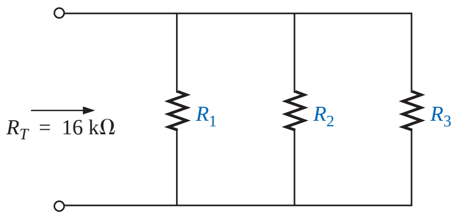

FIG. 6.16
Example 6.9.

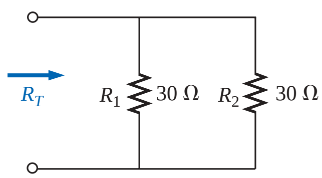

FIG. 6.17
Example 6.10: two equal, parallel resistors.

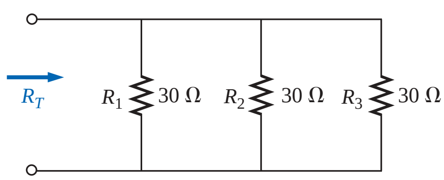

FIG. 6.18
Adding a third parallel resistor of equal value to the network of Fig. 6.17.

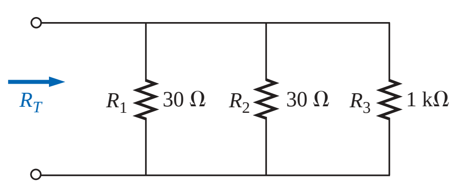

FIG. 6.19
Adding a much larger parallel resistor to the network of Fig. 6.17.

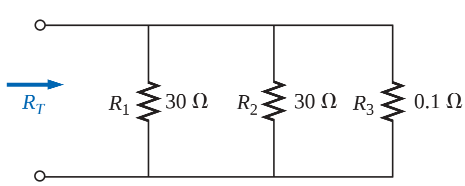

FIG. 6.20
Adding a much smaller parallel resistor to the network of Fig. 6.17.

Substituting values:

$$R_2 = \frac{(9\ \text{k}\Omega)(12\ \text{k}\Omega)}{12\ \text{k}\Omega - 9\ \text{k}\Omega}$$
$$= \frac{108\ \text{k}\Omega}{3} = \mathbf{36\ k\Omega}$$

EXAMPLE 6.9 Determine the values of R_1, R_2, and R_3 in Fig. 6.16 if $R_2 = 2R_1$ and $R_3 = 2R_2$ and the total resistance is 16 kΩ.

Solution:

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$
$$\frac{1}{16\ \text{k}\Omega} = \frac{1}{R_1} + \frac{1}{2R_1} + \frac{1}{4R_1}$$

since $\qquad R_3 = 2R_2 = 2(2R_1) = 4R_1$

and
$$\frac{1}{16\ \text{k}\Omega} = \frac{1}{R_1} + \frac{1}{2}\left(\frac{1}{R_1}\right) + \frac{1}{4}\left(\frac{1}{R_1}\right)$$
$$\frac{1}{16\ \text{k}\Omega} = 1.75\left(\frac{1}{R_1}\right)$$

with $\qquad R_1 = 1.75(16\ \text{k}\Omega) = \mathbf{28\ k\Omega}$

Recall for series circuits that the total resistance will always increase as additional elements are added in series.

For parallel resistors, the total resistance will always decrease as additional elements are added in parallel.

The next example demonstrates this unique characteristic of parallel resistors.

EXAMPLE 6.10

a. Determine the total resistance of the network of Fig. 6.17.
b. What is the effect on the total resistance of the network of Fig. 6.17 if an additional resistor of the same value is added, as shown in Fig. 6.18?
c. What is the effect on the total resistance of the network of Fig. 6.17 if a very large resistance is added in parallel, as shown in Fig. 6.19?
d. What is the effect on the total resistance of the network of Fig. 6.17 if a very small resistance is added in parallel, as shown in Fig. 6.20?

Solutions:

a. $R_T = 30\ \Omega \parallel 30\ \Omega = \dfrac{30\ \Omega}{2} = \mathbf{15\ \Omega}$

b. $R_T = 30\ \Omega \parallel 30\ \Omega \parallel 30\ \Omega = \dfrac{30\ \Omega}{3} = \mathbf{10\ \Omega} < 15\ \Omega$

R_T decreased

c. $R_T = 30\ \Omega \parallel 30\ \Omega \parallel 1\ \text{k}\Omega = 15\ \Omega \parallel 1\ \text{k}\Omega$

$$= \frac{(15\ \Omega)(1000\ \Omega)}{15\ \Omega + 1000\ \Omega} = \mathbf{14.778\ \Omega} < 15\ \Omega$$

Small decrease in R_T

d. $R_T = 30\ \Omega \parallel 30\ \Omega \parallel 0.1\ \Omega = 15\ \Omega \parallel 0.1\ \Omega$

$$= \frac{(15\ \Omega)(0.1\ \Omega)}{15\ \Omega + 0.1\ \Omega} = \mathbf{0.099\ \Omega} < 15\ \Omega$$

Significant decrease in R_T

In each case the total resistance of the network decreased with the increase of an additional parallel resistive element, no matter how large or small. Note also that the total resistance is also smaller than that of the smallest parallel element.

6.4 PARALLEL CIRCUITS

The network of Fig. 6.21 is the simplest of parallel circuits. All the elements have terminals *a* and *b* in common. The total resistance is determined by $R_T = R_1R_2/(R_1 + R_2)$, and the source current by $I_s = E/R_T$. Throughout the text, the subscript *s* will be used to denote a property of the source. Since the terminals of the battery are connected directly across the resistors R_1 and R_2, the following should be obvious:

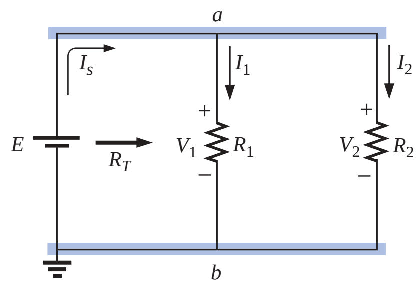

FIG. 6.21
Parallel network.

The voltage across parallel elements is the same.

Using this fact will result in

$$V_1 = V_2 = E$$

and
$$I_1 = \frac{V_1}{R_1} = \frac{E}{R_1}$$

with
$$I_2 = \frac{V_2}{R_2} = \frac{E}{R_2}$$

If we take the equation for the total resistance and multiply both sides by the applied voltage, we obtain

$$E\left(\frac{1}{R_T}\right) = E\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

and
$$\frac{E}{R_T} = \frac{E}{R_1} + \frac{E}{R_2}$$

Substituting the Ohm's law relationships appearing above, we find that the source current

$$I_s = I_1 + I_2$$

permitting the following conclusion:

For single-source parallel networks, the source current (I_s) is equal to the sum of the individual branch currents.

The power dissipated by the resistors and delivered by the source can be determined from

$$P_1 = V_1I_1 = I_1^2R_1 = \frac{V_1^2}{R_1}$$

$$P_2 = V_2I_2 = I_2^2R_2 = \frac{V_2^2}{R_2}$$

$$P_s = EI_s = I_s^2R_T = \frac{E^2}{R_T}$$

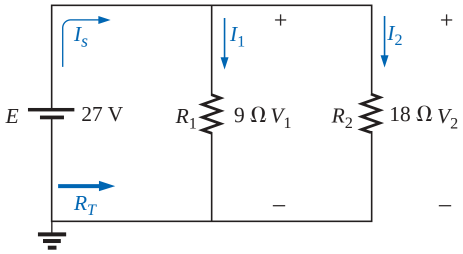

FIG. 6.22
Example 6.11.

EXAMPLE 6.11 For the parallel network of Fig. 6.22:

a. Calculate R_T.
b. Determine I_s.
c. Calculate I_1 and I_2, and demonstrate that $I_s = I_1 + I_2$.
d. Determine the power to each resistive load.
e. Determine the power delivered by the source, and compare it to the total power dissipated by the resistive elements.

Solutions:

a. $R_T = \dfrac{R_1R_2}{R_1 + R_2} = \dfrac{(9\ \Omega)(18\ \Omega)}{9\ \Omega + 18\ \Omega} = \dfrac{162\ \Omega}{27} = \mathbf{6\ \Omega}$

b. $I_s = \dfrac{E}{R_T} = \dfrac{27\text{ V}}{6\ \Omega} = \mathbf{4.5\ A}$

c. $I_1 = \dfrac{V_1}{R_1} = \dfrac{E}{R_1} = \dfrac{27\text{ V}}{9\ \Omega} = \mathbf{3\ A}$

$I_2 = \dfrac{V_2}{R_2} = \dfrac{E}{R_2} = \dfrac{27\text{ V}}{18\ \Omega} = \mathbf{1.5\ A}$

$I_s = I_1 + I_2$

$4.5\text{ A} = 3\text{ A} + 1.5\text{ A}$

$\mathbf{4.5\ A = 4.5\ A}$ (checks)

d. $P_1 = V_1I_1 = EI_1 = (27\text{ V})(3\text{ A}) = \mathbf{81\ W}$
$P_2 = V_2I_2 = EI_2 = (27\text{ V})(1.5\text{ A}) = \mathbf{40.5\ W}$

e. $P_s = EI_s = (27\text{ V})(4.5\text{ A}) = \mathbf{121.5\ W}$
$= P_1 + P_2 = 81\text{ W} + 40.5\text{ W} = \mathbf{121.5\ W}$

EXAMPLE 6.12 Given the information provided in Fig. 6.23:

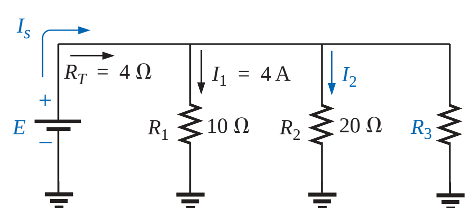

FIG. 6.23
Example 6.12.

a. Determine R_3.
b. Calculate E.
c. Find I_s.
d. Find I_2.
e. Determine P_2.

Solutions:

a. $\dfrac{1}{R_T} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$

$\dfrac{1}{4\ \Omega} = \dfrac{1}{10\ \Omega} + \dfrac{1}{20\ \Omega} + \dfrac{1}{R_3}$

$$0.25\text{ S} = 0.1\text{ S} + 0.05\text{ S} + \frac{1}{R_3}$$

$$0.25\text{ S} = 0.15\text{ S} + \frac{1}{R_3}$$

$$\frac{1}{R_3} = 0.1\text{ S}$$

$$R_3 = \frac{1}{0.1\text{ S}} = \mathbf{10\ \Omega}$$

b. $E = V_1 = I_1R_1 = (4\text{ A})(10\ \Omega) = \mathbf{40\ V}$

c. $I_s = \dfrac{E}{R_T} = \dfrac{40\text{ V}}{4\ \Omega} = \mathbf{10\ A}$

d. $I_2 = \dfrac{V_2}{R_2} = \dfrac{E}{R_2} = \dfrac{40\text{ V}}{20\ \Omega} = \mathbf{2\ A}$

e. $P_2 = I_2^2R_2 = (2\text{ A})^2(20\ \Omega) = \mathbf{80\ W}$

Mathcad Solution: This example provides an excellent opportunity to practice our skills using Mathcad. As shown in Fig. 6.24, the known parameters and quantities of the network are entered first, followed by an equation for the unknown resistor R_3. Note that after the first division operator was selected, a left bracket was established (to be followed eventually by a right enclosure bracket) to tell the computer that the mathematical operations in the denominator must be carried out first before the division into 1. In addition, each individual division into 1 is separated by brackets to ensure that the division operation is performed before each quantity is added to the neighboring factor. Finally, keep in mind that the Mathcad bracket must encompass each individual expression of the denominator before you place the right bracket in place.

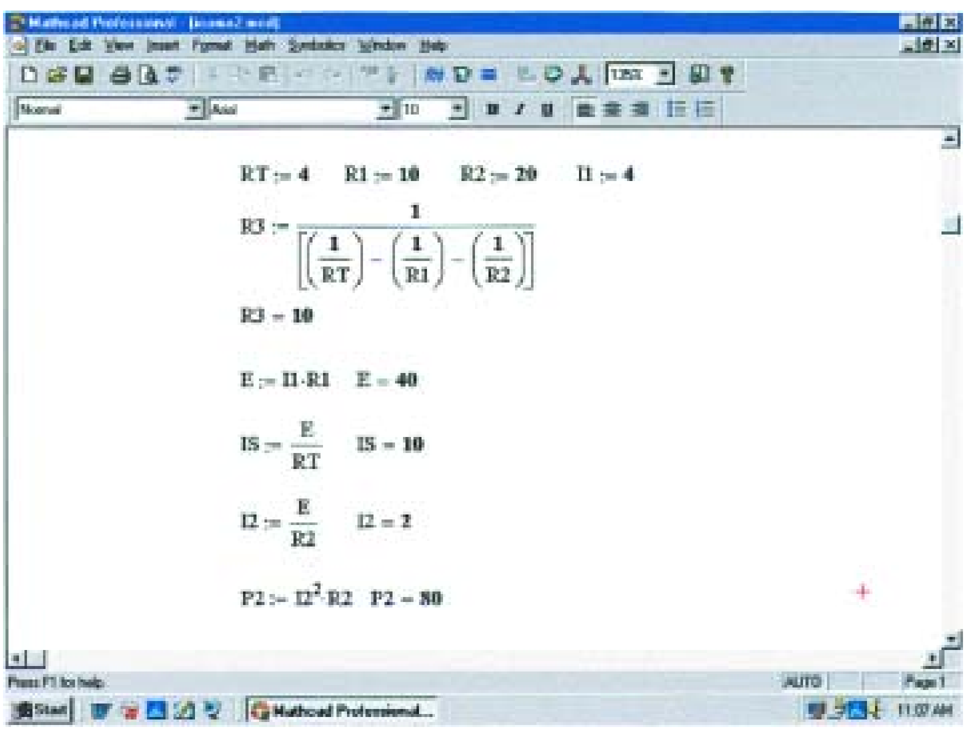

FIG. 6.24

Using Mathcad to confirm the results of Example 6.12.

In each case, the quantity of interest was entered below the defining equation to obtain the numerical result by selecting an equal sign. As expected, all the results match the longhand solution.

6.5 KIRCHHOFF'S CURRENT LAW

Kirchhoff's voltage law provides an important relationship among voltage levels around any closed loop of a network. We now consider **Kirchhoff's current law (KCL),** which provides an equally important relationship among current levels at any junction.

Kirchhoff's current law (KCL) states that the algebraic sum of the currents entering and leaving an area, system, or junction is zero.

In other words,

the sum of the currents entering an area, system, or junction must equal the sum of the currents leaving the area, system, or junction.

In equation form:

$$\Sigma I_{\text{entering}} = \Sigma I_{\text{leaving}} \tag{6.8}$$

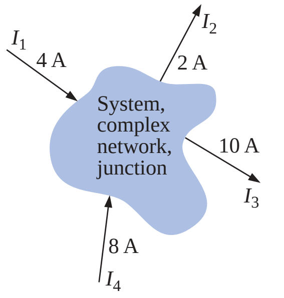

FIG. 6.25
Introducing Kirchhoff's current law.

In Fig. 6.25, for instance, the shaded area can enclose an entire system, a complex network, or simply a junction of two or more paths. In each case the current entering must equal that leaving, as witnessed by the fact that

$$I_1 + I_4 = I_2 + I_3$$
$$4\text{ A} + 8\text{ A} = 2\text{ A} + 10\text{ A}$$
$$12\text{ A} = 12\text{ A}$$

The most common application of the law will be at the junction of two or more paths of current flow, as shown in Fig. 6.26. For some students it is difficult initially to determine whether a current is entering or leaving a junction. One approach that may help is to picture yourself as standing on the junction and treating the path currents as arrows. If the arrow appears to be heading toward you, as is the case for I_1 in Fig. 6.26, then it is entering the junction. If you see the tail of the arrow (from the junction) as it travels down its path away from you, it is leaving the junction, as is the case for I_2 and I_3 in Fig. 6.26.

Applying Kirchhoff's current law to the junction of Fig. 6.26:

$$\Sigma I_{\text{entering}} = \Sigma I_{\text{leaving}}$$
$$6\text{ A} = 2\text{ A} + 4\text{ A}$$
$$6\text{ A} = 6\text{ A} \quad \text{(checks)}$$

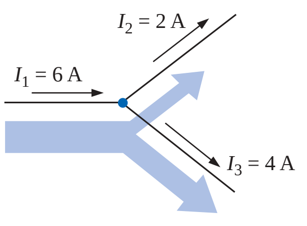

FIG. 6.26
Demonstrating Kirchhoff's current law.

In the next two examples, unknown currents can be determined by applying Kirchhoff's current law. Simply remember to place all current levels entering a junction to the left of the equals sign and the sum of all currents leaving a junction to the right of the equals sign. The water-in-the-pipe analogy is an excellent one for supporting and clarifying the preceding law. Quite obviously, the sum total of the water entering a junction must equal the total of the water leaving the exit pipes.

In technology the term **node** is commonly used to refer to a junction of two or more branches. Therefore, this term will be used frequently in the analyses that follow.

EXAMPLE 6.13 Determine the currents I_3 and I_4 of Fig. 6.27 using Kirchhoff's current law.

Solution: We must first work with junction a since the only unknown is I_3. At junction b there are two unknowns, and both cannot be determined from one application of the law.

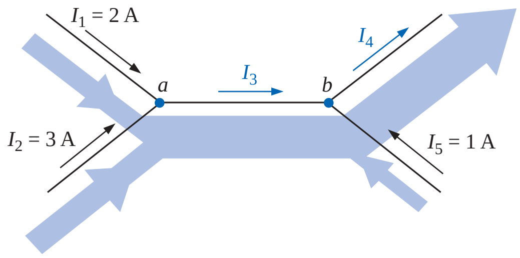

FIG. 6.27
Example 6.13.

At a:

$$\begin{aligned}\Sigma I_{\text{entering}} &= \Sigma I_{\text{leaving}}\\ I_1 + I_2 &= I_3\\ 2\text{ A} + 3\text{ A} &= I_3\\ I_3 &= \mathbf{5\text{ A}}\end{aligned}$$

At b:

$$\begin{aligned}\Sigma I_{\text{entering}} &= \Sigma I_{\text{leaving}}\\ I_3 + I_5 &= I_4\\ 5\text{ A} + 1\text{ A} &= I_4\\ I_4 &= \mathbf{6\text{ A}}\end{aligned}$$

EXAMPLE 6.14 Determine I_1, I_3, I_4, and I_5 for the network of Fig. 6.28.

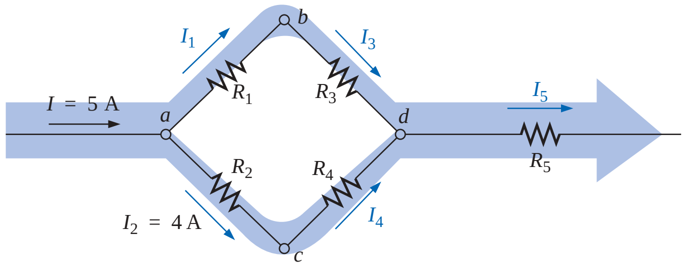

FIG. 6.28
Example 6.14.

Solution: At a:

$$\begin{aligned}\Sigma I_{\text{entering}} &= \Sigma I_{\text{leaving}}\\ I &= I_1 + I_2\\ 5\text{ A} &= I_1 + 4\text{ A}\end{aligned}$$

Subtracting 4 A from both sides gives

$$\begin{aligned}5\text{ A} - 4\text{ A} &= I_1 + 4\cancel{\text{A}} - 4\cancel{\text{A}}\\ I_1 &= 5\text{ A} - 4\text{ A} = \mathbf{1\text{ A}}\end{aligned}$$

At b:

$$\Sigma I_{\text{entering}} = \Sigma I_{\text{leaving}}$$
$$I_1 = I_3 = \mathbf{1\,A}$$

as it should, since R_1 and R_3 are in series and the current is the same in series elements.

At c:

$$I_2 = I_4 = \mathbf{4\,A}$$

for the same reasons given for junction b.

At d:

$$\Sigma I_{\text{entering}} = \Sigma I_{\text{leaving}}$$
$$I_3 + I_4 = I_5$$
$$1\,\text{A} + 4\,\text{A} = I_5$$
$$I_5 = \mathbf{5\,A}$$

If we enclose the entire network, we find that the current entering is $I = 5$ A; the net current leaving from the far right is $I_5 = 5$ A. The two must be equal since the net current entering any system must equal that leaving.

EXAMPLE 6.15 Determine the currents I_3 and I_5 of Fig. 6.29 through applications of Kirchhoff's current law.

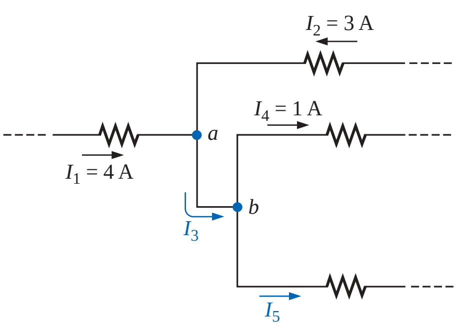

FIG. 6.29
Example 6.15.

Solution: Note that since node b has two unknown quantities and node a has only one, we must first apply Kirchhoff's current law to node a. The result can then be applied to node b.

For node a,

$$I_1 + I_2 = I_3$$
$$4\,\text{A} + 3\,\text{A} = I_3$$

and $$I_3 = \mathbf{7\,A}$$

For node b,

$$I_3 = I_4 + I_5$$
$$7\,\text{A} = 1\,\text{A} + I_5$$

and $$I_5 = 7\,\text{A} - 1\,\text{A} = \mathbf{6\,A}$$

EXAMPLE 6.16 Find the magnitude and direction of the currents I_3, I_4, I_6, and I_7 for the network of Fig. 6.30. Even though the elements are not in series or parallel, Kirchhoff's current law can be applied to determine all the unknown currents.

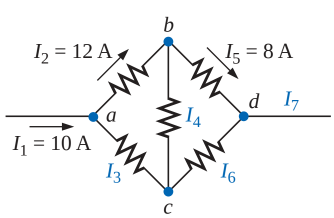

FIG. 6.30
Example 6.16.

Solution: Considering the overall system, we know that the current entering must equal that leaving. Therefore,

$$I_7 = I_1 = \mathbf{10\,A}$$

Since 10 A are entering node a and 12 A are leaving, I_3 must be supplying current to the node.

Applying Kirchhoff's current law at node a,

$$I_1 + I_3 = I_2$$
$$10\,\text{A} + I_3 = 12\,\text{A}$$

and $$I_3 = 12\,\text{A} - 10\,\text{A} = \mathbf{2\,A}$$

At node b, since 12 A are entering and 8 A are leaving, I_4 must be leaving. Therefore,

$$I_2 = I_4 + I_5$$
$$12\text{ A} = I_4 + 8\text{ A}$$
and
$$I_4 = 12\text{ A} - 8\text{ A} = \mathbf{4\text{ A}}$$

At node *c*, I_3 is leaving at 2 A and I_4 is entering at 4 A, requiring that I_6 be leaving. Applying Kirchhoff's current law at node *c*,

$$I_4 = I_3 + I_6$$
$$4\text{ A} = 2\text{ A} + I_6$$
and
$$I_6 = 4\text{ A} - 2\text{ A} = \mathbf{2\text{ A}}$$

As a check at node *d*,

$$I_5 + I_6 = I_7$$
$$8\text{ A} + 2\text{ A} = 10\text{ A}$$
$$\mathbf{10\text{ A} = 10\text{ A}} \quad \text{(checks)}$$

Looking back at Example 6.11, we find that the current entering the top node is 4.5 A and the current leaving the node is $I_1 + I_2 = 3\text{ A} + 1.5\text{ A} = 4.5\text{ A}$. For Example 6.12, we have

$$I_s = I_1 + I_2 + I_3$$
$$10\text{ A} = 4\text{ A} + 2\text{ A} + I_3$$
and
$$I_3 = 10\text{ A} - 6\text{ A} = 4\text{ A}$$

The application of Kirchhoff's current law is not limited to networks where all the internal connections are known or visible. For instance, all the currents of the integrated circuit of Fig. 6.31 are known except I_1. By treating the system as a single node, we can apply Kirchhoff's current law using the following values to ensure an accurate listing of all known quantities:

I_i	I_o
10 mA	5 mA
4 mA	4 mA
8 mA	2 mA
22 mA	6 mA
	17 mA

Noting the total input current versus that leaving clearly reveals that I_1 is a current of 22 mA − 17 mA = 5 mA leaving the system.

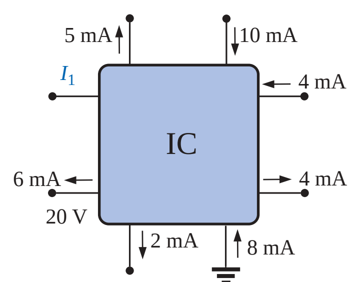

FIG. 6.31
Integrated circuit.

6.6 CURRENT DIVIDER RULE

As the name suggests, the **current divider rule (CDR)** will determine how the current entering a set of parallel branches will split between the elements.

For two parallel elements of equal value, the current will divide equally.

For parallel elements with different values, the smaller the resistance, the greater the share of input current.

For parallel elements of different values, the current will split with a ratio equal to the inverse of their resistor values.

For example, if one of two parallel resistors is twice the other, then the current through the larger resistor will be half the other.

In Fig. 6.32, since I_1 is 1 mA and R_1 is six times R_3, the current through R_3 must be 6 mA (without making any other calculations including the total current or the actual resistance levels). For R_2 the current must be 2 mA since R_1 is twice R_2. The total current must then be the sum of I_1, I_2, and I_3, or 9 mA. In total, therefore, knowing only the current through R_1, we were able to find all the other currents of the configuration without knowing anything more about the network.

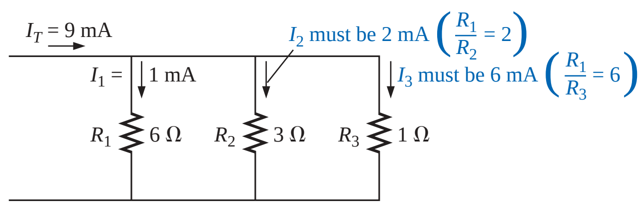

FIG. 6.32
Demonstrating how current will divide between unequal resistors.

For networks in which only the resistor values are given along with the input current, the current divider rule should be applied to determine the various branch currents. It can be derived using the network of Fig. 6.33.

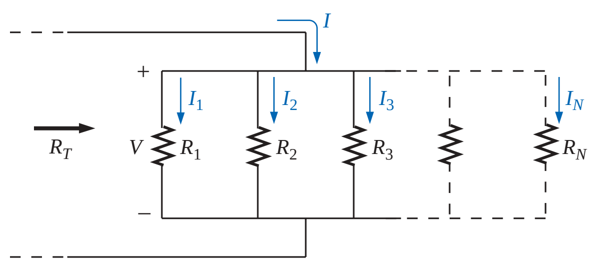

FIG. 6.33
Deriving the current divider rule.

The input current I equals V/R_T, where R_T is the total resistance of the parallel branches. Substituting $V = I_x R_x$ into the above equation, where I_x refers to the current through a parallel branch of resistance R_x, we have

$$I = \frac{V}{R_T} = \frac{I_x R_x}{R_T}$$

and

$$I_x = \frac{R_T}{R_x} I \qquad (6.9)$$

which is the general form for the current divider rule. In words, the current through any parallel branch is equal to the product of the *total* resistance of the parallel branches and the input current divided by the resistance of the branch through which the current is to be determined.

For the current I_1,

$$I_1 = \frac{R_T}{R_1} I$$

and for I_2,

$$I_2 = \frac{R_T}{R_2} I$$

and so on.

For the particular case of *two parallel resistors,* as shown in Fig. 6.34,

$$R_T = \frac{R_1 R_2}{R_1 + R_2}$$

and

$$I_1 = \frac{R_T}{R_1} I = \frac{\dfrac{R_1 R_2}{R_1 + R_2}}{R_1} I$$

and

Note difference in subscripts.

$$I_1 = \frac{R_2 I}{R_1 + R_2} \tag{6.10}$$

Similarly for I_2,

$$I_2 = \frac{R_1 I}{R_1 + R_2} \tag{6.11}$$

In words, for two parallel branches, the current through either branch is equal to the product of the *other* parallel resistor and the input current divided by the *sum* (not the total parallel resistance) of the two parallel resistances.

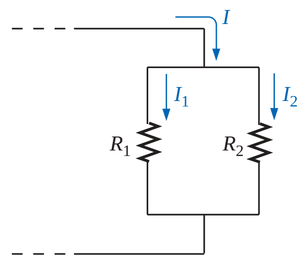

FIG. 6.34
Developing an equation for current division between two parallel resistors.

EXAMPLE 6.17 Determine the current I_2 for the network of Fig. 6.35 using the current divider rule.

Solution:

$$I_2 = \frac{R_1 I_s}{R_1 + R_2} = \frac{(4\ \text{k}\Omega)(6\ \text{A})}{4\ \text{k}\Omega + 8\ \text{k}\Omega} = \frac{4}{12}(6\ \text{A}) = \frac{1}{3}(6\ \text{A})$$
$$= \mathbf{2\ A}$$

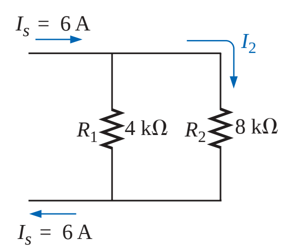

FIG. 6.35
Example 6.17.

EXAMPLE 6.18 Find the current I_1 for the network of Fig. 6.36.

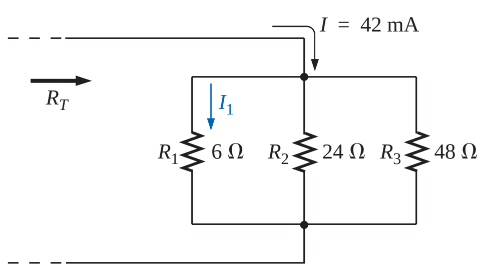

FIG. 6.36
Example 6.18

Solution: There are two options for solving this problem. The first is to use Eq. (6.9) as follows:

$$\frac{1}{R_T} = \frac{1}{6\ \Omega} + \frac{1}{24\ \Omega} + \frac{1}{48\ \Omega} = 0.1667\ \text{S} + 0.0417\ \text{S} + 0.0208\ \text{S}$$
$$= 0.2292\ \text{S}$$

and $$R_T = \frac{1}{0.2292 \text{ S}} = 4.363\ \Omega$$

with $$I_1 = \frac{R_T}{R_1} I = \frac{4.363\ \Omega}{6\ \Omega}(42 \text{ mA}) = \mathbf{30.54\ mA}$$

The second option is to apply Eq. (6.10) once after combining R_2 and R_3 as follows:

$$24\ \Omega \parallel 48\ \Omega = \frac{(24\ \Omega)(48\ \Omega)}{24\ \Omega + 48\ \Omega} = 16\ \Omega$$

and $$I_1 = \frac{16\ \Omega(42 \text{ mA})}{16\ \Omega + 6\ \Omega} = \mathbf{30.54\ mA}$$

Both options generated the same answer, leaving you with a choice for future calculations involving more than two parallel resistors.

EXAMPLE 6.19 Determine the magnitude of the currents I_1, I_2, and I_3 for the network of Fig. 6.37.

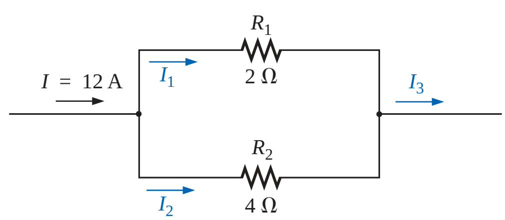

FIG. 6.37
Example 6.19.

Solution: By Eq. (6.10), the current divider rule,

$$I_1 = \frac{R_2 I}{R_1 + R_2} = \frac{(4\ \Omega)(12 \text{ A})}{2\ \Omega + 4\ \Omega} = \mathbf{8\ A}$$

Applying Kirchhoff's current law,

$$I = I_1 + I_2$$

and $$I_2 = I - I_1 = 12 \text{ A} - 8 \text{ A} = \mathbf{4\ A}$$

or, using the current divider rule again,

$$I_2 = \frac{R_1 I}{R_1 + R_2} = \frac{(2\ \Omega)(12 \text{ A})}{2\ \Omega + 4\ \Omega} = \mathbf{4\ A}$$

The total current entering the parallel branches must equal that leaving. Therefore,

$$I_3 = I = \mathbf{12\ A}$$

or $$I_3 = I_1 + I_2 = 8 \text{ A} + 4 \text{ A} = \mathbf{12\ A}$$

EXAMPLE 6.20 Determine the resistance R_1 to effect the division of current in Fig. 6.38.

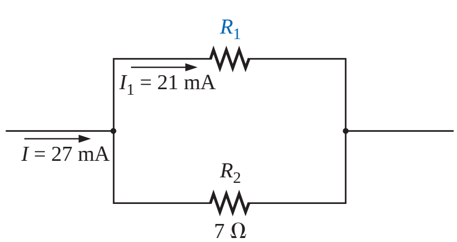

FIG. 6.38
Example 6.20.

Solution: Applying the current divider rule,

$$I_1 = \frac{R_2 I}{R_1 + R_2}$$

and
$$(R_1 + R_2)I_1 = R_2 I$$
$$R_1 I_1 + R_2 I_1 = R_2 I$$
$$R_1 I_1 = R_2 I - R_2 I_1$$
$$R_1 = \frac{R_2(I - I_1)}{I_1}$$

Substituting values:

$$R_1 = \frac{7\ \Omega(27\text{ mA} - 21\text{ mA})}{21\text{ mA}}$$
$$= 7\ \Omega\left(\frac{6}{21}\right) = \frac{42\ \Omega}{21} = \mathbf{2\ \Omega}$$

An alternative approach is

$$I_2 = I - I_1 \quad \text{(Kirchhoff's current law)}$$
$$= 27\text{ mA} - 21\text{ mA} = 6\text{ mA}$$
$$V_2 = I_2 R_2 = (6\text{ mA})(7\ \Omega) = 42\text{ mV}$$
$$V_1 = I_1 R_1 = V_2 = 42\text{ mV}$$

and
$$R_1 = \frac{V_1}{I_1} = \frac{42\text{ mV}}{21\text{ mA}} = \mathbf{2\ \Omega}$$

From the examples just described, note the following:

Current seeks the path of least resistance.

That is,

1. More current passes through the smaller of two parallel resistors.
2. The current entering any number of parallel resistors divides into these resistors as the inverse ratio of their ohmic values. This relationship is depicted in Fig. 6.39.

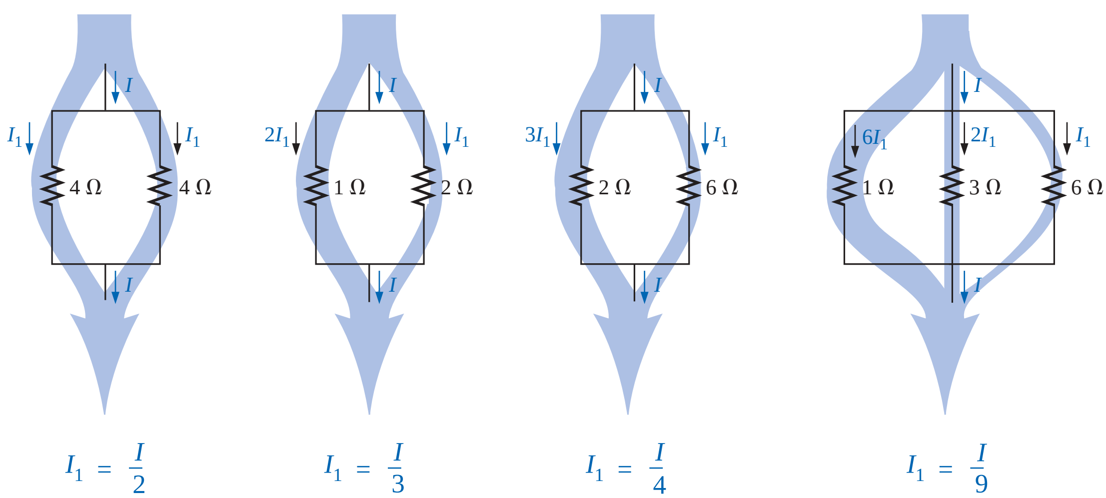

FIG. 6.39
Current division through parallel branches.

6.7 VOLTAGE SOURCES IN PARALLEL

Voltage sources are placed in parallel as shown in Fig. 6.40 only if they have the same voltage rating. The primary reason for placing two or more batteries in parallel of the same terminal voltage would be to

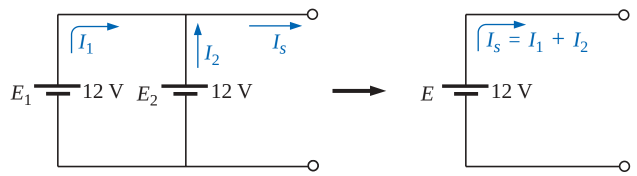

FIG. 6.40
Parallel voltage sources.

increase the current rating (and, therefore, the power rating) of the source. As shown in Fig. 6.40, the current rating of the combination is determined by $I_s = I_1 + I_2$ at the same terminal voltage. The resulting power rating is twice that available with one supply.

If two batteries of different terminal voltages were placed in parallel, both would be left ineffective or damaged because the terminal voltage of the larger battery would try to drop rapidly to that of the lower supply. Consider two lead-acid car batteries of different terminal voltage placed in parallel, as shown in Fig. 6.41.

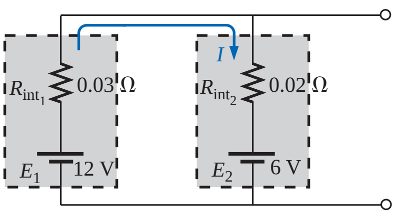

FIG. 6.41
Parallel batteries of different terminal voltages.

The relatively small internal resistances of the batteries are the only current-limiting elements of the resulting series circuit. The current is

$$I = \frac{E_1 - E_2}{R_{int_1} + R_{int_2}} = \frac{12\text{ V} - 6\text{ V}}{0.03\ \Omega + 0.02\ \Omega} = \frac{6\text{ V}}{0.05\ \Omega} = \mathbf{120\ A}$$

which far exceeds the continuous drain rating of the larger supply, resulting in a rapid discharge of E_1 and a destructive impact on the smaller supply.

6.8 OPEN AND SHORT CIRCUITS

Open circuits and short circuits can often cause more confusion and difficulty in the analysis of a system than standard series or parallel configurations. This will become more obvious in the chapters to follow when we apply some of the methods and theorems.

An **open circuit** is simply two isolated terminals not connected by an element of any kind, as shown in Fig. 6.42(a). Since a path for conduction does not exist, the current associated with an open circuit must always be zero. The voltage across the open circuit, however, can be any value, as determined by the system it is connected to. In summary, therefore,

an open circuit can have a potential difference (voltage) across its terminals, but the current is always zero amperes.

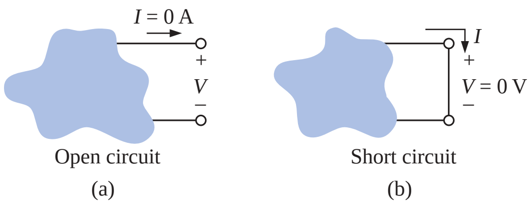

FIG. 6.42
Two special network configurations.

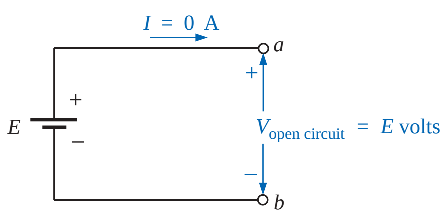

FIG. 6.43
Demonstrating the characteristics of an open circuit.

In Fig. 6.43, an open circuit exists between terminals *a* and *b*. As shown in the figure, the voltage across the open-circuit terminals is the supply voltage, but the current is zero due to the absence of a complete circuit.

A **short circuit** is a very low resistance, direct connection between two terminals of a network, as shown in Fig. 6.42(b). The current through the short circuit can be any value, as determined by the system it is connected to, but the voltage across the short circuit will always be zero volts because the resistance of the short circuit is assumed to be essentially zero ohms and $V = IR = I(0\ \Omega) = 0$ V.

In summary, therefore,

a short circuit can carry a current of a level determined by the external circuit, but the potential difference (voltage) across its terminals is always zero volts.

In Fig. 6.44(a), the current through the 2-Ω resistor is 5 A. If a short circuit should develop across the 2-Ω resistor, the total resistance of the parallel combination of the 2-Ω resistor and the short (of essentially zero ohms) will be $2\ \Omega \parallel 0\ \Omega = \dfrac{(2\ \Omega)(0\ \Omega)}{2\ \Omega + 0\ \Omega} = 0\ \Omega$, and the current will rise to very high levels, as determined by Ohm's law:

$$I = \frac{E}{R} = \frac{10\text{ V}}{0\ \Omega} \longrightarrow \infty\text{ A}$$

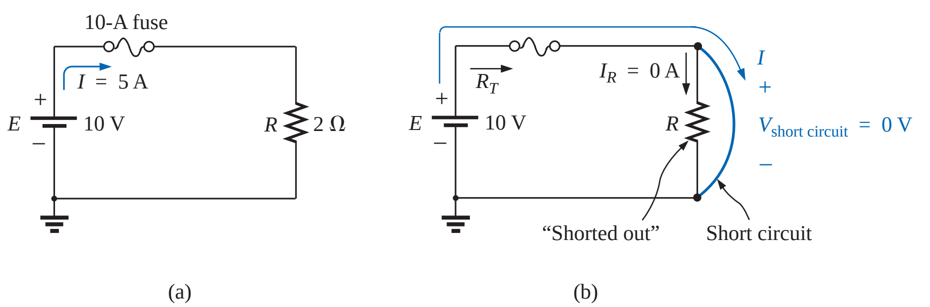

FIG. 6.44
Demonstrating the effect of a short circuit on current levels.

The effect of the 2-Ω resistor has effectively been "shorted out" by the low-resistance connection. The maximum current is now limited only by the circuit breaker or fuse in series with the source.

For the layperson, the terminology *short circuit* or *open circuit* is usually associated with dire situations such as power loss, smoke, or fire. However, in network analysis both can play an integral role in determining specific parameters about a system. Most often, however, if a short-circuit condition is to be established, it is accomplished with a *jumper*—a lead of negligible resistance to be connected between the points of interest. Establishing an open circuit simply requires making sure that the terminals of interest are isolated from each other.

EXAMPLE 6.21 Determine the voltage V_{ab} for the network of Fig. 6.45.

Solution: The open circuit requires that I be zero amperes. The voltage drop across both resistors is therefore zero volts since $V = IR = (0)R = 0$ V. Applying Kirchhoff's voltage law around the closed loop,

$$V_{ab} = E = \mathbf{20\ V}$$

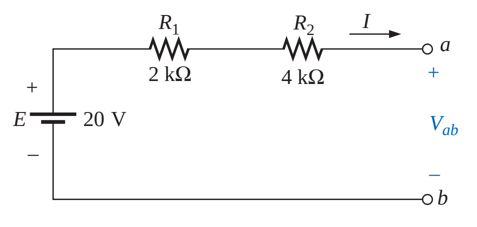

FIG. 6.45
Example 6.21.

EXAMPLE 6.22 Determine the voltages V_{ab} and V_{cd} for the network of Fig. 6.46.

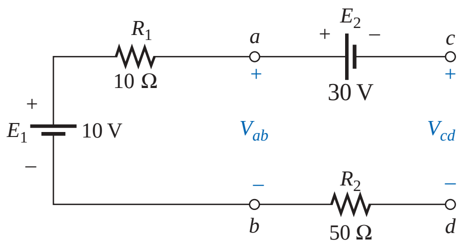

FIG. 6.46
Example 6.22.

Solution: The current through the system is zero amperes due to the open circuit, resulting in a 0-V drop across each resistor. Both resistors can therefore be replaced by short circuits, as shown in Fig. 6.47. The voltage V_{ab} is then directly across the 10-V battery, and

$$V_{ab} = E_1 = \mathbf{10\ V}$$

The voltage V_{cd} requires an application of Kirchhoff's voltage law:

$$+E_1 - E_2 - V_{cd} = 0$$

or $$V_{cd} = E_1 - E_2 = 10\text{ V} - 30\text{ V} = \mathbf{-20\ V}$$

The negative sign in the solution simply indicates that the actual voltage V_{cd} has the opposite polarity of that appearing in Fig. 6.46.

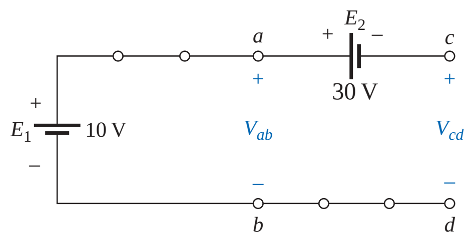

FIG. 6.47
Circuit of Fig. 6.46 redrawn.

EXAMPLE 6.23 Determine the unknown voltage and current for each network of Fig. 6.48.

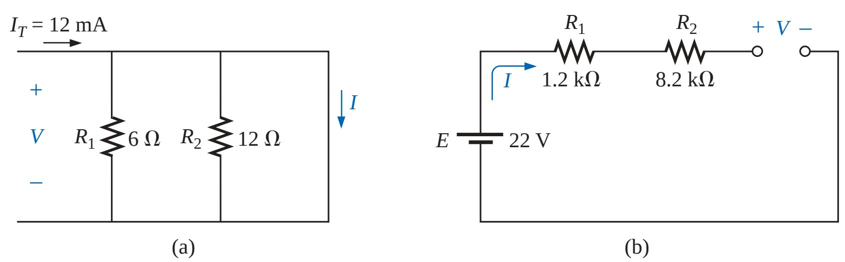

FIG. 6.48
Example 6.23.

Solution: For the network of Fig. 6.48(a), the current I_T will take the path of least resistance, and, since the short-circuit condition at the end of the network is the least-resistance path, all the current will pass through the short circuit. This conclusion can be verified using Eq. (6.9). The voltage across the network is the same as that across the short circuit and will be zero volts, as shown in Fig. 6.49(a).

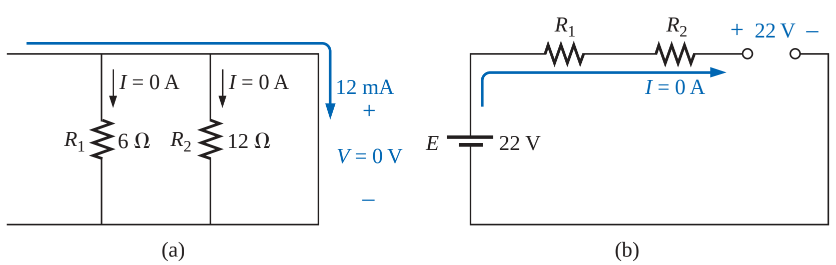

FIG. 6.49
Solutions to Example 6.23.

For the network of Fig. 6.48(b), the open-circuit condition requires that the current be zero amperes. The voltage drops across the resistors must therefore be zero volts, as determined by Ohm's law [$V_R = IR = (0)R = 0$ V], with the resistors simply acting as a connection from the supply to the open circuit. The result is that the open-circuit voltage will be $E = 22$ V, as shown in Fig. 6.49(b).

EXAMPLE 6.24 Calculate the current I and the voltage V for the network of Fig. 6.50.

Solution: The 10-kΩ resistor has been effectively shorted out by the jumper, resulting in the equivalent network of Fig. 6.51. Using Ohm's law,

$$I = \frac{E}{R_1} = \frac{18\text{ V}}{5\text{ k}\Omega} = \mathbf{3.6\text{ mA}}$$

and

$$V = E = \mathbf{18\text{ V}}$$

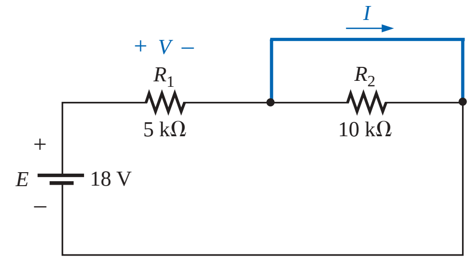

FIG. 6.50
Example 6.24.

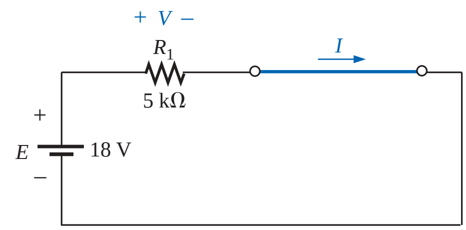

FIG. 6.51
Network of Fig. 6.50 redrawn.

EXAMPLE 6.25 Determine V and I for the network of Fig. 6.52 if the resistor R_2 is shorted out.

Solution: The redrawn network appears in Fig. 6.53. The current through the 3-Ω resistor is zero due to the open circuit, causing all the current I to pass through the jumper. Since $V_{3\Omega} = IR = (0)R = 0$ V, the voltage V is directly across the short, and

$$V = \mathbf{0\text{ V}}$$

with

$$I = \frac{E}{R_1} = \frac{6\text{ V}}{2\ \Omega} = \mathbf{3\text{ A}}$$

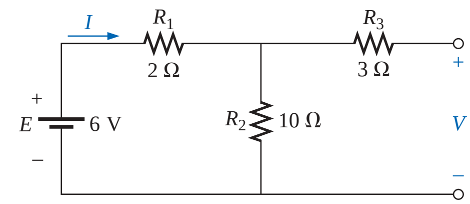

FIG. 6.52
Example 6.25.

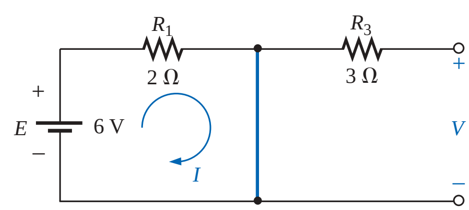

FIG. 6.53
Network of Fig. 6.52 with R_2 replaced by a jumper.

6.9 VOLTMETERS: LOADING EFFECT

In Chapters 2 and 5, it was noted that voltmeters are always placed across an element to measure the potential difference. We now realize that this connection is synonymous with placing the voltmeter in parallel with the element. The insertion of a meter in parallel with a resistor results in a combination of parallel resistors as shown in Fig. 6.54. Since the resistance of two parallel branches is always less than the smaller parallel resistance, the resistance of the voltmeter should be as large as possible (ideally infinite). In Fig. 6.54, a DMM with an internal resistance of 11 MΩ is measuring the voltage across a 10-kΩ resistor. The total resistance of the combination is

$$R_T = 10\text{ k}\Omega \parallel 11\text{ M}\Omega = \frac{(10^4\ \Omega)(11 \times 10^6\ \Omega)}{10^4\ \Omega + (11 \times 10^6\ \Omega)} = 9.99\text{ k}\Omega$$

and we find that the network is essentially undisturbed. However, if we use a VOM with an internal resistance of 50 kΩ on the 2.5-V scale, the parallel resistance is

$$R_T = 10\text{ k}\Omega \parallel 50\text{ k}\Omega = \frac{(10^4\ \Omega)(50 \times 10^3\ \Omega)}{10^4\ \Omega + (50 \times 10^3\ \Omega)} = 8.33\text{ k}\Omega$$

and the behavior of the network will be altered somewhat since the 10-kΩ resistor will now appear to be 8.33 kΩ to the rest of the network.

The loading of a network by the insertion of meters is not to be taken lightly, especially in research efforts where accuracy is a primary consideration. It is good practice always to check the meter resistance level

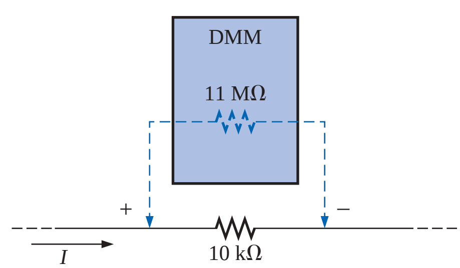

FIG. 6.54
Voltmeter loading.