Phase Relations

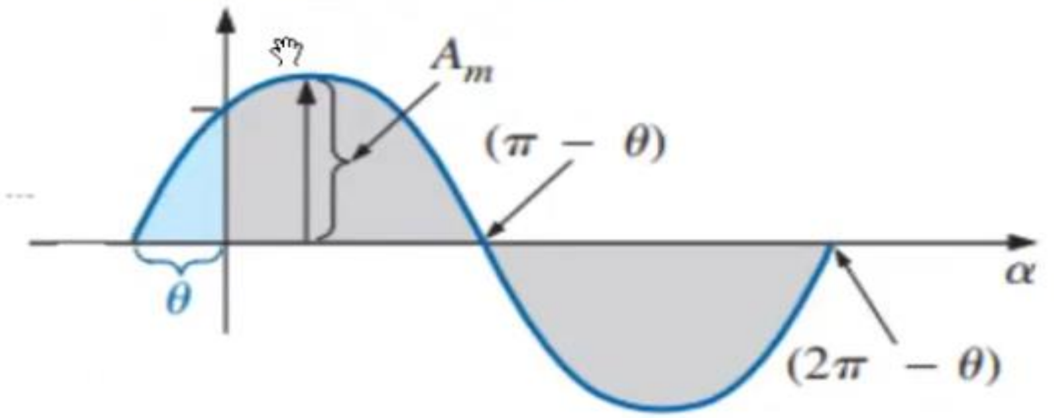
If the sine waveform is shifted to the right or left of θ°, the expression becomes

$$A_m \sin(\omega t \pm \theta)$$

Where θ is the angle in degrees or radians that the waveform has been shifted

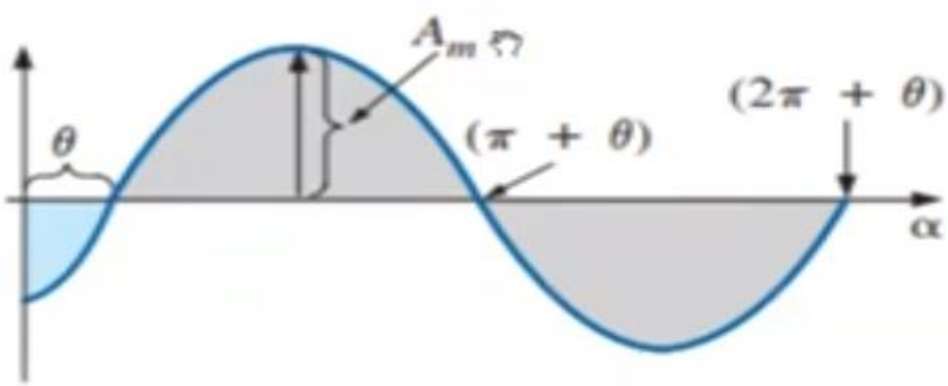
If the waveform passes through the horizontal axis before θ, as shown in figure below, the expression is:

$$A_m \sin(\omega t + \theta)$$

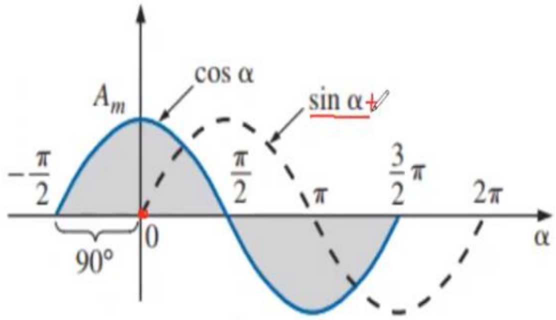


If the waveform passes through the horizontal axis before θ, as shown in figure below, the expression is:

$$A_m \sin(\omega t - \theta)$$



- If $\theta = \frac{\pi}{2}$ *or* 90^o ,then the waveform as shown in Figure below, it is called a *cosine wave;* that is,



$$\sin(\omega t + 90^\circ) = \sin\left(\omega t + \frac{\pi}{2}\right) = \cos \omega t$$

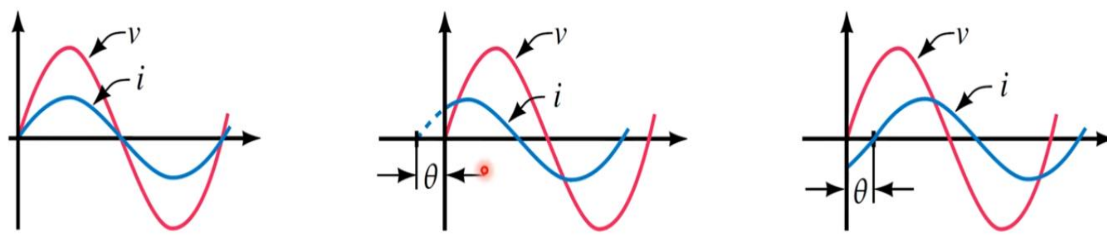
$$\sin \omega t = \cos(\omega t - 90^\circ) = \cos\left(\omega t - \frac{\pi}{2}\right)$$

Note:

- the cosine curve is said to *lead* the sine curve by 90°
- the sine curve is said to *lag* the cosine curve by 90°.
- The 90° is referred to as the ***phase angle*** between the two waveforms.
- The two waveforms are ***out of phase*** by 90°
- If both waveforms cross the axis at the same point with the same slope, they are ***in phase***.

Phase Difference

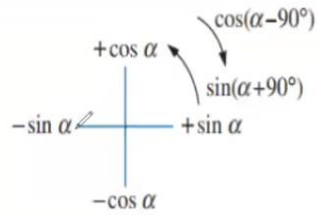
Phase difference refers to the angular displacement between different waveforms of the same frequency.



(a) In phase (b) Current leads (c) Current lags

There are some other useful relations:

$$\cos\alpha = \sin(\alpha + 90°)$$
$$\sin\alpha = \cos(\alpha - 90°)$$
$$-\sin\alpha = \sin(\alpha \pm 180°)$$
$$-\cos\alpha = \sin(\alpha + 270°) = \sin(\alpha - 90°)$$



$$\sin(-\alpha) = -\sin\alpha$$
$$\cos(-\alpha) = \cos\alpha$$

Ex: given the sinusoidal $(t) = -125\cos(200\pi t - 260)$, express the time domain equation as a positive sinusoidal function.

Sol: $v(t) = -125\cos(200\pi t - 260)$

$$v(t) = 125\cos(200\pi t - 260 + 180)$$

$$v(t) = 125\cos(200\pi t - 80)$$

and $cos(wt) = \sin(wt + 90)$

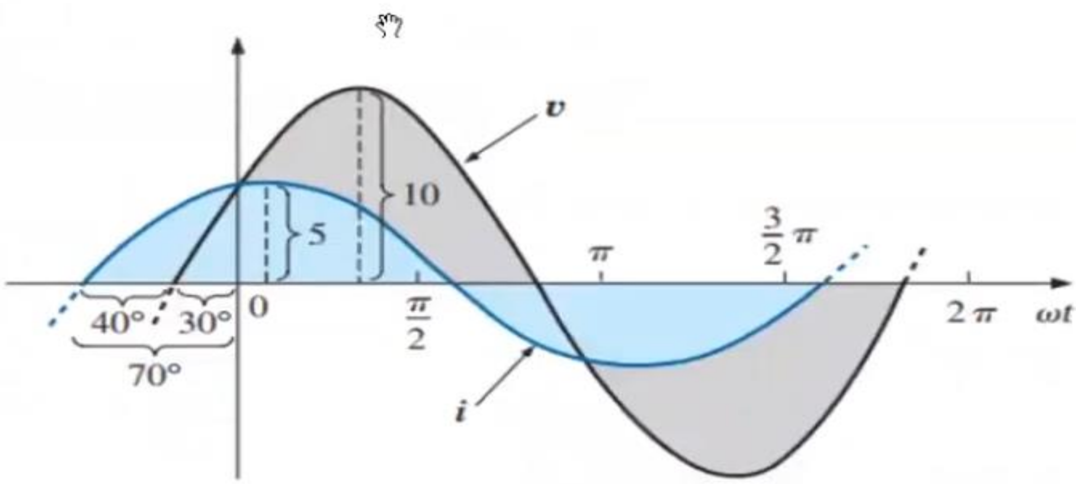
then $v(t) = 125\sin(200\pi t - 80 + 90) = 125\sin(200\pi t + 10)$

Ex: What is the phase relationship between the sinusoidal waveforms of each of the following sets?

a. $v = 10\sin(\omega t + 30°)$
$i = 5\sin(\omega t + 70°)$

Sol:

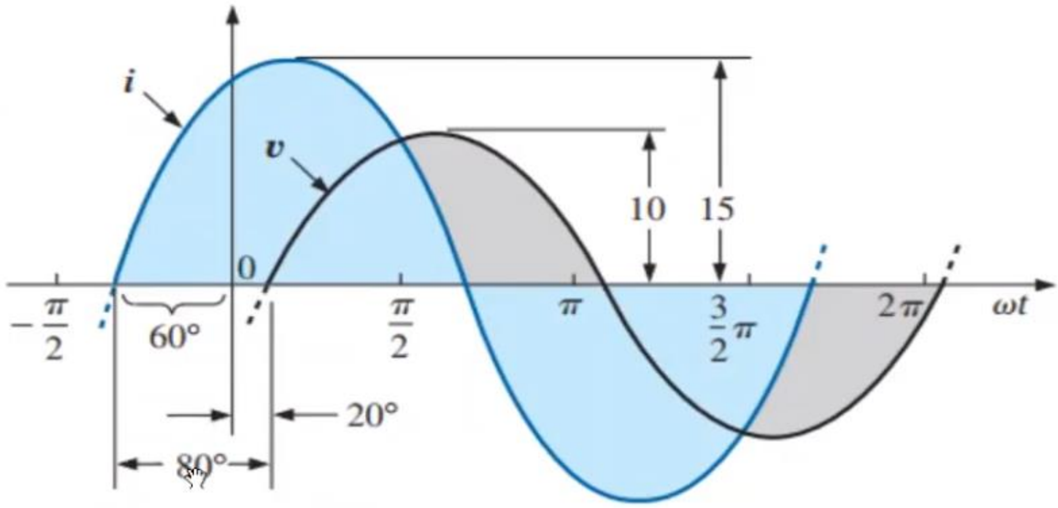
***i* leads *v* by 40º, or *v* lags *i* by 40º.**



b. $i = 15 \sin(\omega t + 60°)$
$v = 10 \sin(\omega t - 20°)$

Sol:

***i* leads *v* by 80º, or *v* lags *i* by 80º.**

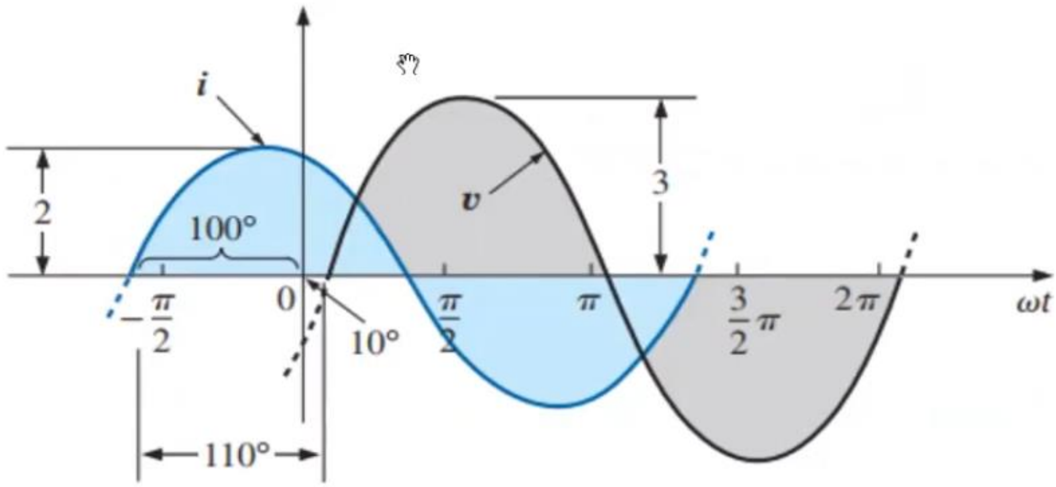


c. $i = 2 \cos(\omega t + 10°)$
$v = 3 \sin(\omega t - 10°)$

Sol:

$$i = 2 \cos(\omega t + 10°) = 2 \sin(\omega t + 10° + 90°)$$
$$= 2 \sin(\omega t + 100°)$$

***i* leads *v* by 110º, or *v* lags *i* by 110º**



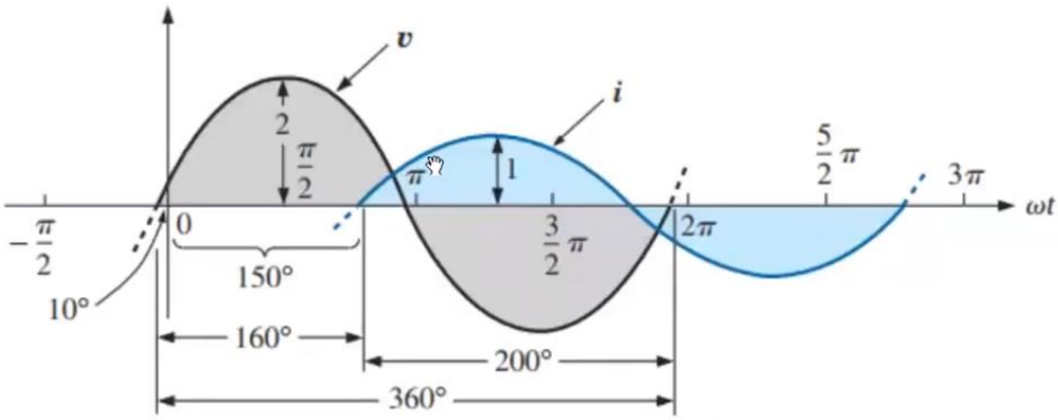
d. $i = -\sin(\omega t + 30°)$
$v = 2\sin(\omega t + 10°)$

Sol:

$$-\sin(\omega t + 30°) = \sin(\omega t + 30° - 180°) = \sin(\omega t - 150°)$$

Note: $-180°$

***v* leads *i* by 160º, or *i* lags *v* by 160º.**

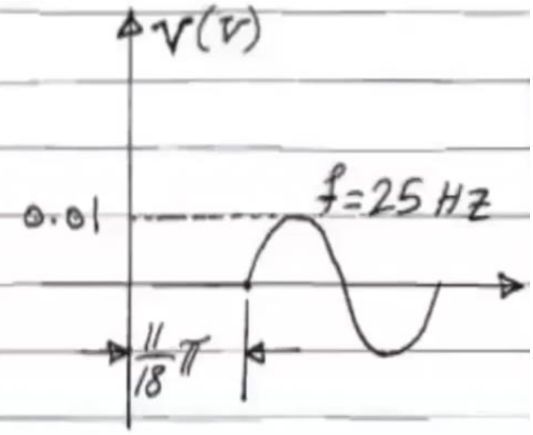


e. $i = -2\cos(\omega t - 60°)$
$v = 3\sin(\omega t - 150°)$

Ex: write the analytical expression for the waveforms in the following figures with phase in degree.

Sol:

a)



$v = E_m \sin(wt \mp \theta^\circ)$

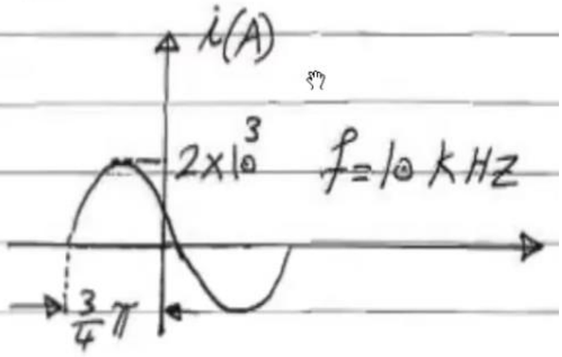
$\theta^\circ = \frac{180^\circ}{\pi} * \frac{11}{18}\pi = 110^\circ$

$W = 2\pi f = 157 \text{ rad/s}$

$W = 2\pi f = 157 \text{ rad/s}$

$\therefore V = 0.01 \sin(157t - 110^\circ)$

b)



$W = 2\pi f = 2\pi * 10k$

$W = 62.8 * 10^3 \text{ r/s}$

$\theta^\circ = \frac{180^\circ}{\pi} * \frac{3}{4}\pi = 135^\circ$

$\therefore i = I_m \sin(wt + \theta^\circ)$

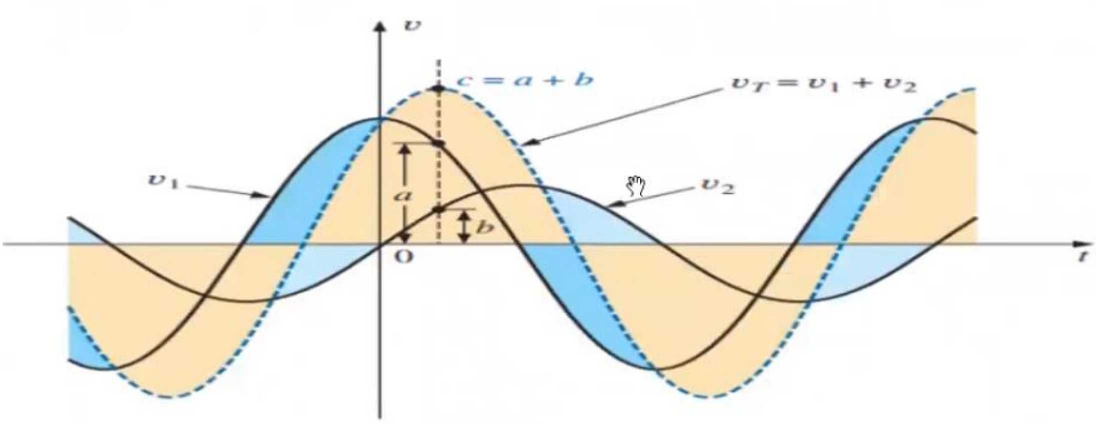
$\therefore i = 2 * 10^3 \sin(62.8 * 10^3 t + 135^\circ)$

PHASORS

The addition of sinusoidal voltages and currents is frequently required in the analysis of Ac circuits. One lengthy but valid of performing this operation is to place both sinusoidal waveform on the same set of axes and add algebraically the magnitude of each at every point along the abscissa, as shown for

c =a + b in figure below

This, however, can be a long and tedious process with limited accuracy



A shorter method uses the rotating radius vector. This radius vector, having a constant magnitude (length) with one end fixed at the origin, is called a phasor when applied to electric circuits.

In general, for all of the analysis to follow the phasor form of a sinusoidal voltage or current, will be

$$v = V_m \sin(\omega t \pm \theta) \Rightarrow V_{eff} \angle \pm \theta$$

$$i = I_m \sin(\omega t \pm \theta) \Rightarrow I_{eff} \angle \pm \theta$$

Where V_{eff} and I_{eff} are the effective values

$$V_{eff} = \frac{V_m}{\sqrt{2}} = 0.707\, V_m \;;\quad I_{eff} = \frac{I_m}{\sqrt{2}} = 0.707 I_m \;;\ \theta \text{ is the phase angle}$$

Ex: Convert the following from the time to the phasor domain:

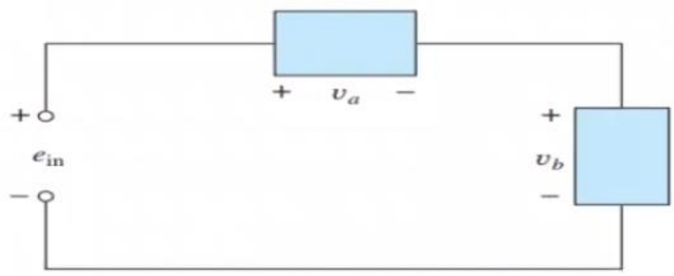
Time Domain	Phasor Domain
a. $\sqrt{2}(50)\sin \omega t$	$\mathbf{50\ \angle 0^\circ}$
b. $69.6\sin(\omega t + 72^\circ)$	$(0.707)(69.6)\ \angle 72^\circ = \mathbf{49.21\ \angle 72^\circ}$
c. $45\cos \omega t$	$(0.707)(45)\ \angle 90^\circ = \mathbf{31.82\ \angle 90^\circ}$

Ex: Write the sinusoidal expression for the following phasors if the frequency is 60 Hz:

Phasor Domain	Time Domain
a. $\mathbf{I} = 10\ \angle 30^\circ$	$i = \sqrt{2}(10)\sin(2\pi 60t + 30^\circ)$ and $i = \mathbf{14.14\sin(377t + 30^\circ)}$
b. $\mathbf{V} = 115\ \angle -70^\circ$	$v = \sqrt{2}(115)\sin(377t - 70^\circ)$ and $v = \mathbf{162.6\sin(377t - 70^\circ)}$

Ex: Find the input voltage of the circuit in Figure below if :

$$\left.\begin{aligned} v_a &= 50\sin(377t + 30^\circ) \\ v_b &= 30\sin(377t + 60^\circ) \end{aligned}\right\} f = 60\text{ Hz}$$



Solution: Applying Kirchhoff's voltage law, we have

$$e_{in} = v_a + v_b$$

Converting from the time to the phasor domain yields

$$v_a = 50\sin(377t + 30^\circ) \Rightarrow \mathbf{V}_a = 35.35\text{ V}\angle 30^\circ$$
$$v_b = 30\sin(377t + 60^\circ) \Rightarrow \mathbf{V}_b = 21.21\text{ V}\angle 60^\circ$$

Converting from polar to rectangular form for addition yields

$$\mathbf{V}_a = 35.35\text{ V}\angle 30^\circ = 30.61\text{ V} + j17.68\text{ V}$$
$$\mathbf{V}_b = 21.21\text{ V}\angle 60^\circ = 10.61\text{ V} + j18.37\text{ V}$$

Then

$$\mathbf{E}_{in} = \mathbf{V}_a + \mathbf{V}_b = (30.61\text{ V} + j17.68\text{ V}) + (10.61\text{ V} + j18.37\text{ V})$$
$$= 41.22\text{ V} + j36.05\text{ V}$$

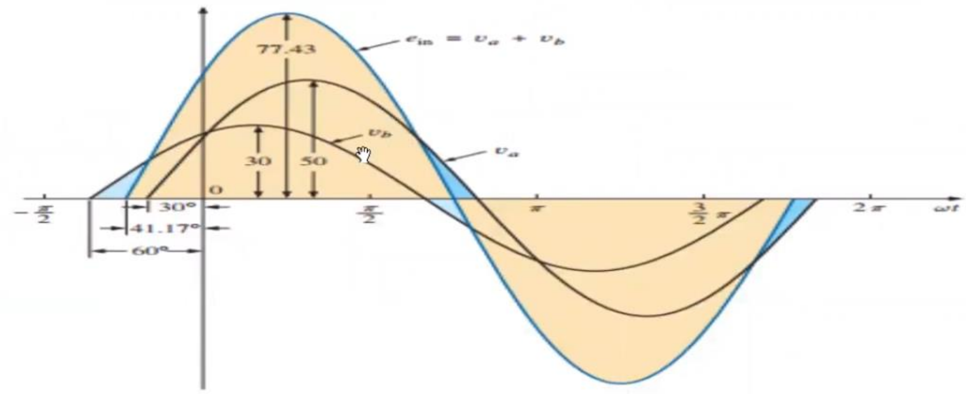
Converting from rectangular to polar form, we have

$$\mathbf{E}_{in} = 41.22\text{ V} + j36.05\text{ V} = 54.76\text{ V}\angle 41.17^\circ$$

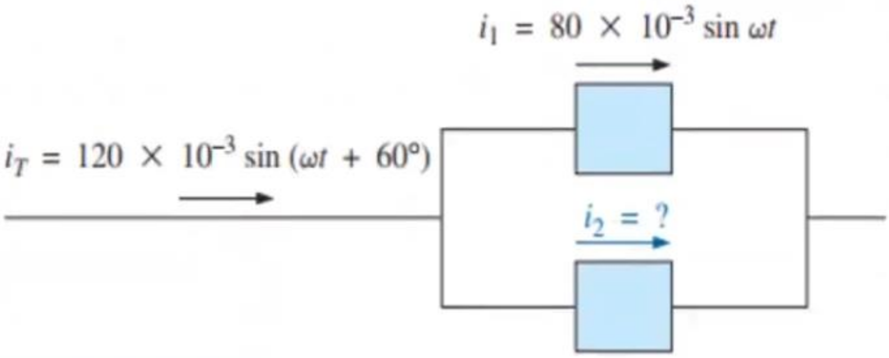
Converting from the phasor to the time domain, we obtain

$$\mathbf{E}_{in} = 54.76\text{ V}\angle 41.17^\circ \Rightarrow e_{in} = \sqrt{2}(54.76)\sin(377t + 41.17^\circ)$$

and $$e_{in} = \mathbf{77.43\sin(377t + 41.17^\circ)}$$



Ex: Determine the current i_2 for the network in Figure below



Solution: Applying Kirchhoff's current law, we obtain

$$i_T = i_1 + i_2 \quad \text{or} \quad i_2 = i_T - i_1$$

Converting from the time to the phasor domain yields

$$i_T = 120 \times 10^{-3} \sin(\omega t + 60°) \Rightarrow 84.84 \text{ mA} \angle 60°$$
$$i_1 = 80 \times 10^{-3} \sin \omega t \Rightarrow 56.56 \text{ mA} \angle 0°$$

Converting from polar to rectangular form for subtraction yields

$$\mathbf{I}_T = 84.84 \text{ mA} \angle 60° = 42.42 \text{ mA} + j73.47 \text{ mA}$$
$$\mathbf{I}_1 = 56.56 \text{ mA} \angle 0° = 56.56 \text{ mA} + j0$$

Then

$$\mathbf{I}_2 = \mathbf{I}_T - \mathbf{I}_1$$
$$= (42.42 \text{ mA} + j73.47 \text{ mA}) - (56.56 \text{ mA} + j0)$$

and
$$\mathbf{I}_2 = -14.14 \text{ mA} + j73.47 \text{ mA}$$

Converting from rectangular to polar form, we have

$$\mathbf{I}_2 = 74.82 \text{ mA} \angle 100.89°$$

Converting from the phasor to the time domain, we have

$$\mathbf{I}_2 = 74.82 \text{ mA} \angle 100.89° \Rightarrow$$
$$i_2 = \sqrt{2}(74.82 \times 10^{-3}) \sin(\omega t + 100.89°)$$

and
$$i_2 = \mathbf{105.8 \times 10^{-3} \sin(\omega t + 100.89°)}$$