Complex Numbers

This lecture introduce a system of **complex numbers** that, when related to the sinusoidal ac waveform, results in a technique for finding the algebraic sum of sinusoidal waveforms that is quick, direct, and accurate.

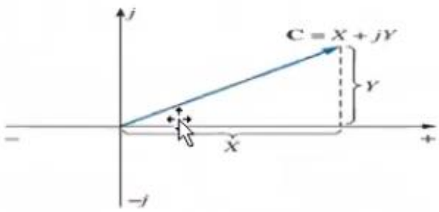
A **complex number** represents a point in a two-dimensional plane located with reference to two distinct axes. This point can also determine a radius vector drawn from the origin to the point. The horizontal axis is called the ***real* axis** or ***resistance axis***, while the vertical axis is called the ***imaginary* axis** or ***reactance axis***. Both are labeled in Figure below:

Two forms are used to represent a complex number: ***rectangular*** and ***polar***. Each can represent a point in the plane or a radius vector drawn from the origin to that point.

RECTANGULAR FORM

The format for the **rectangular form** is

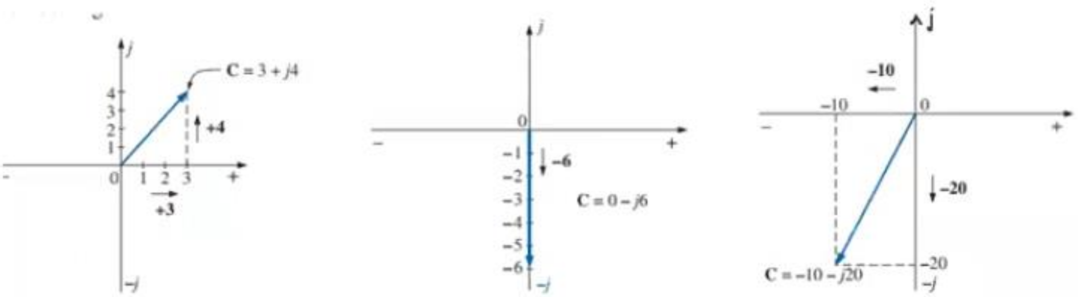
$$\mathbf{C} = X + jY$$



EXAMPLE Sketch the following complex numbers in the complex plane:

a. $\mathbf{C} = 3 + j4$
b. $\mathbf{C} = 0 - j6$
c. $\mathbf{C} = -10 - j20$

Solution:

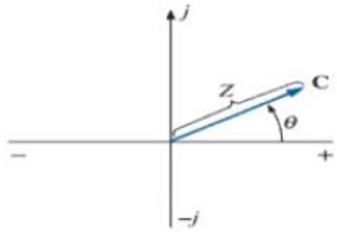


POLAR FORM

The format for the **polar form** is

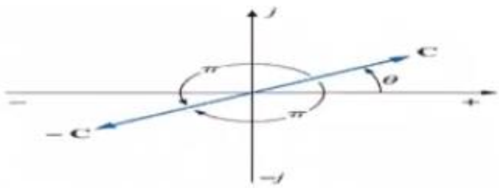
$$\mathbf{C} = Z \angle \theta$$

Z indicates magnitude only and ***θ*** is *always measured* ***counterclockwise*** *(CCW) from the positive real axis,* as shown in Figure below. Angles measured in the ***clockwise*** direction from the positive real axis must have a negative sign associated with them.



A negative sign in front of the polar form has the effect shown in Figure below. Note that it results in a complex number directly opposite the complex number with a positive sign.

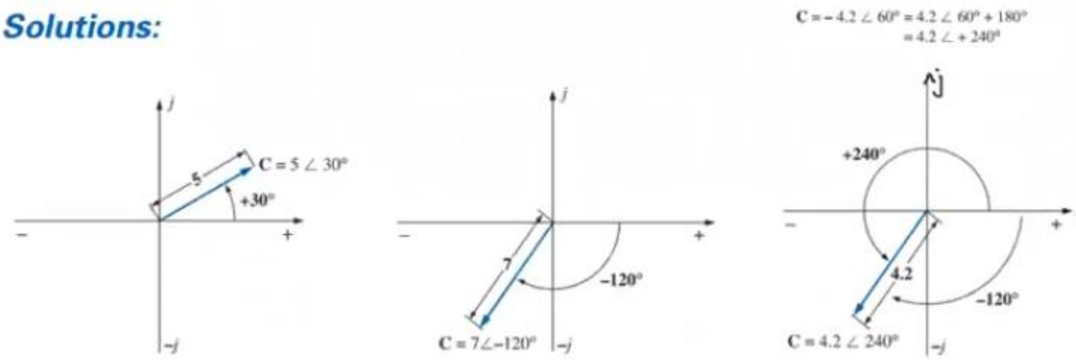
$$-\mathbf{C} = -Z \angle \theta = Z \angle \theta \pm 180°$$



EXAMPLE Sketch the following complex numbers in the complex plane:

a. $\mathbf{C} = 5 \angle 30°$
b. $\mathbf{C} = 7 \angle -120°$
c. $\mathbf{C} = -4.2 \angle 60°$

Solutions:



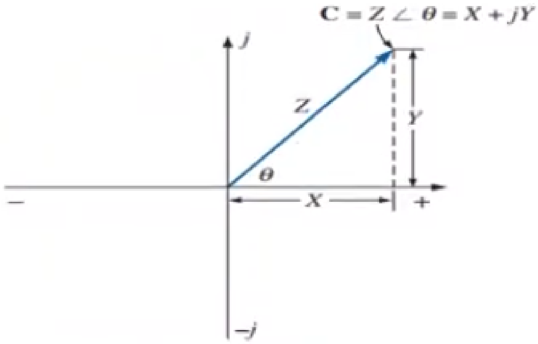
CONVERSION BETWEEN FORMS

The two forms are related by the following equations.

Rectangular to Polar

$$Z = \sqrt{X^2 + Y^2}$$

$$\theta = \tan^{-1}\frac{Y}{X}$$



Polar to Rectangular

$$X = Z\cos\theta$$

$$Y = Z\sin\theta$$

EXAMPLE Convert the following from rectangular to polar form:

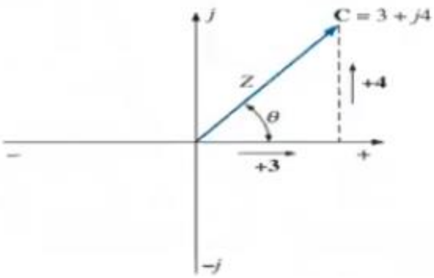
$$\mathbf{C} = 3 + j4$$

Solution:

$$Z = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^\circ$$

and $$\mathbf{C} = 5\angle 53.13^\circ$$



EXAMPLE Convert the following from polar to rectangular form:

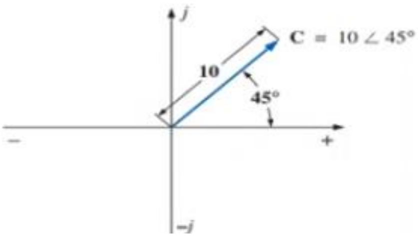
$$\mathbf{C} = 10\angle 45^\circ$$

Solution:

$$X = 10\cos 45^\circ = (10)(0.707) = 7.07$$

$$Y = 10\sin 45^\circ = (10)(0.707) = 7.07$$

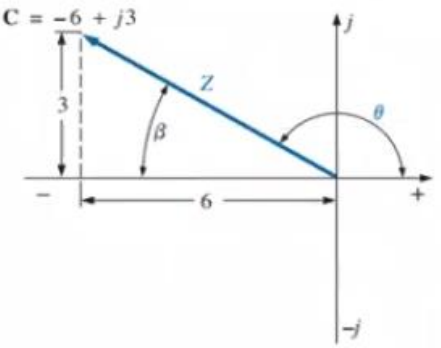
and $$\mathbf{C} = 7.07 + j7.07$$



EXAMPLE Convert the following from rectangular to polar form:

$$\mathbf{C} = -6 + j3$$

Solution:



$$Z = \sqrt{(6)^2 + (3)^2} = \sqrt{45} = 6.71$$

$$\beta = \tan^{-1}\left(\frac{3}{6}\right) = 26.57°$$

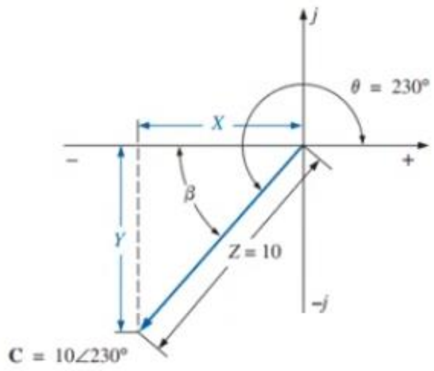
$$\theta = 180° - 26.57° = 153.43°$$

and $$\mathbf{C} = \mathbf{6.71 \angle 153.43°}$$

EXAMPLE Convert the following from polar to rectangular form:

$$\mathbf{C} = 10 \angle 230°$$

Solution:



$$X = Z\cos\beta = 10\cos(230° - 180°) = 10\cos 50°$$

$$= (10)(0.6428) = 6.428$$

$$Y = Z\sin\beta = 10\sin 50° = (10)(0.7660) = 7.660$$

and $$\mathbf{C} = \mathbf{-6.43 - j7.66}$$

MATHEMATICAL OPERATIONS WITH COMPLEX NUMBERS

Complex numbers lend themselves readily to the basic mathematical operations of *addition, subtraction, multiplication, and division*. A few basic rules and definitions must be understood before considering these operations. Let us first examine the symbol ***j*** associated with imaginary numbers. By definition,

$$j = \sqrt{-1}$$

Thus, $$j^2 = -1$$

and $$j^3 = j^2 j = -1j = -j$$

with $$j^4 = j^2 j^2 = (-1)(-1) = +1$$

$$j^5 = j$$

and so on. Further,

$$\frac{1}{j} = (1)\left(\frac{1}{j}\right) = \left(\frac{j}{j}\right)\left(\frac{1}{j}\right) = \frac{j}{j^2} = \frac{j}{-1}$$

and $$\frac{1}{j} = -j$$

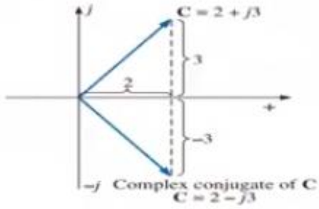
Complex Conjugate

The **conjugate** or **complex conjugate** of a complex number can be found by simply changing the sign of the imaginary part in the rectangular form or by using the negative of the angle of the polar form. For example, the conjugate of

$$\mathbf{C} = 2 + j3$$

is

$$2 - j3$$

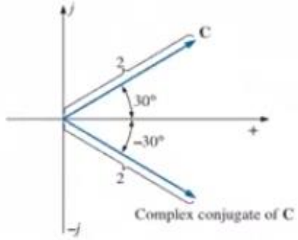


The conjugate of

$$\mathbf{C} = 2 \angle 30^\circ$$

is

$$2 \angle -30^\circ$$



Reciprocal

The **reciprocal** of a complex number is 1 divided by the complex number. For example, the reciprocal of

$$\mathbf{C} = X + jY$$

is

$$\frac{1}{X + jY}$$

and of $Z \angle \theta$,

$$\frac{1}{Z \angle \theta}$$

Addition

To add two or more complex numbers, add the real and imaginary parts separately. For example, if

$$\mathbf{C}_1 = \pm X_1 \pm jY_1 \quad \text{and} \quad \mathbf{C}_2 = \pm X_2 \pm jY_2$$

then

$$\mathbf{C}_1 + \mathbf{C}_2 = (\pm X_1 \pm X_2) + j(\pm Y_1 \pm Y_2)$$

EXAMPLE Add $\mathbf{C}_1 = 2 + j4$ and $\mathbf{C}_2 = 3 + j1$

Solutions: $\mathbf{C}_1 + \mathbf{C}_2 = (2 + 3) + j(4 + 1) = \mathbf{5 + j5}$

EXAMPLE $\mathbf{C}_1 = 3 + j6$ and $\mathbf{C}_2 = -6 + j3$.

Subtraction

In subtraction, the real and imaginary parts are again considered separately. For example, if

$$\mathbf{C}_1 = \pm X_1 \pm jY_1 \quad \text{and} \quad \mathbf{C}_2 = \pm X_2 \pm jY_2$$

then

$$\mathbf{C}_1 - \mathbf{C}_2 = [\pm X_1 - (\pm X_2)] + j[\pm Y_1 - (\pm Y_2)]$$

EXAMPLE Subtract $\mathbf{C}_2 = 1 + j4$ from $\mathbf{C}_1 = 4 + j6$.

Solutions: $\mathbf{C}_1 - \mathbf{C}_2 = (4 - 1) + j(6 - 4) = \mathbf{3 + j2}$

Addition or subtraction cannot be performed in polar form unless the complex numbers have the same angle θ or unless they differ only by multiples of 180°.

EXAMPLE

a. $2\angle 45^\circ + 3\angle 45^\circ = \mathbf{5\angle 45^\circ}$

b. $2\angle 0^\circ - 4\angle 180^\circ = \mathbf{6\angle 0^\circ}$.

Multiplication

To multiply two complex numbers in *rectangular* form, multiply the real and imaginary parts of one in turn by the real and imaginary parts of the other. For example, if

$$\mathbf{C}_1 = X_1 + jY_1 \quad \text{and} \quad \mathbf{C}_2 = X_2 + jY_2$$

$$\mathbf{C}_1 \cdot \mathbf{C}_2 = (X_1X_2 - Y_1Y_2) + j(Y_1X_2 + X_1Y_2)$$

Find $\mathbf{C}_1 \cdot \mathbf{C}_2$ if $\quad \mathbf{C}_1 = 2 + j3 \quad$ and $\quad \mathbf{C}_2 = 5 + j10$

Solutions:

$$\begin{aligned}\mathbf{C}_1 \cdot \mathbf{C}_2 &= [(2)(5) - (3)(10)] + j[(3)(5) + (2)(10)] \\ &= \mathbf{-20 + j35}\end{aligned}$$

In *polar* form, the magnitudes are multiplied and the angles added algebraically. For example, for

$$\mathbf{C}_1 = Z_1 \angle\theta_1 \quad \text{and} \quad \mathbf{C}_2 = Z_2 \angle\theta_2$$

we write

$$\mathbf{C}_1 \cdot \mathbf{C}_2 = Z_1Z_2 \angle\theta_1 + \theta_2$$

Find $\mathbf{C}_1 \cdot \mathbf{C}_2$ if $\quad \mathbf{C}_1 = 5 \angle 20^\circ \quad$ and $\quad \mathbf{C}_2 = 10 \angle 30^\circ$

Solutions: $= (5)(10) \angle 20^\circ + 30^\circ = \mathbf{50 \angle 50^\circ}$

To multiply a complex number in rectangular form by a real number requires that both the real part and the imaginary part be multiplied by the real number. For example,

$$(10)(2 + j3) = 20 + j30$$

and

$$50 \angle 0^\circ\ (0 + j6) = j300 = 300 \angle 90^\circ$$

Division

To divide two complex numbers in *rectangular* form, multiply the numerator and denominator by the conjugate of the denominator and the resulting real and imaginary parts collected. That is, if

$$\mathbf{C}_1 = X_1 + jY_1 \quad \text{and} \quad \mathbf{C}_2 = X_2 + jY_2$$

then

$$\begin{aligned}\frac{\mathbf{C}_1}{\mathbf{C}_2} &= \frac{(X_1 + jY_1)(X_2 - jY_2)}{(X_2 + jY_2)(X_2 - jY_2)} \\ &= \frac{(X_1X_2 + Y_1Y_2) + j(X_2Y_1 - X_1Y_2)}{X_2^2 + Y_2^2}\end{aligned}$$

and

$$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{X_1X_2 + Y_1Y_2}{X_2^2 + Y_2^2} + j\frac{X_2Y_1 - X_1Y_2}{X_2^2 + Y_2^2}$$

The equation does not have to be memorized if the steps above used to obtain it are employed. That is, first multiply the numerator by the complex conjugate of the denominator and separate the real and imaginary terms. Then divide each term by the sum of each term of the denominator squared.

Find $\mathbf{C}_1/\mathbf{C}_2$ if $\mathbf{C}_1 = 1 + j4$ and $\mathbf{C}_2 = 4 + j5$.

To divide a complex number in rectangular form by a real number, both the real part and the imaginary part must be divided by the real number. For example,

$$\frac{8 + j10}{2} = 4 + j5$$

and

$$\frac{6.8 - j0}{2} = 3.4 - j0 = 3.4\angle 0°$$

In *polar* form, division is accomplished by dividing the magnitude of the numerator by the magnitude of the denominator and subtracting the angle of the denominator from that of the numerator. That is, for

$$\mathbf{C}_1 = Z_1 \angle\theta_1 \quad \text{and} \quad \mathbf{C}_2 = Z_2 \angle\theta_2$$

we write

$$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{Z_1}{Z_2}\angle\theta_1 - \theta_2$$

EXAMPLE

a. Find $\mathbf{C}_1/\mathbf{C}_2$ if $\mathbf{C}_1 = 15 \angle 10°$ and $\mathbf{C}_2 = 2 \angle 7°$.
b. Find $\mathbf{C}_1/\mathbf{C}_2$ if $\mathbf{C}_1 = 8 \angle 120°$ and $\mathbf{C}_2 = 16 \angle -50°$.

Solutions:

a. $$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{15 \angle 10°}{2 \angle 7°} = \frac{15}{2} \angle 10° - 7° = \mathbf{7.5 \angle 3°}$$

b. $$\frac{\mathbf{C}_1}{\mathbf{C}_2} = \frac{8 \angle 120°}{16 \angle -50°} = \frac{8}{16} \angle 120° - (-50°) = \mathbf{0.5 \angle 170°}$$

and

$$\frac{1}{X + jY} = \frac{X}{X^2 + Y^2} - j\frac{Y}{X^2 + Y^2}$$

In polar form, the reciprocal is

$$\frac{1}{Z \angle \theta} = \frac{1}{Z} \angle -\theta$$

We obtain the *reciprocal* in the rectangular form by multiplying the numerator and denominator by the complex conjugate of the denominator:

$$\frac{1}{X + jY} = \left(\frac{1}{X + jY}\right)\left(\frac{X - jY}{X - jY}\right) = \frac{X - jY}{X^2 + Y^2}$$

EXAMPLE Perform the following operations, leaving the answer in polar or rectangular form:

a. $\frac{(2 + j3) + (4 + j6)}{(7 + j7) - (3 - j3)} = \frac{(2 + 4) + j(3 + 6)}{(7 - 3) + j(7 + 3)}$

$= \frac{(6 + j9)(4 - j10)}{(4 + j10)(4 - j10)}$

$= \frac{[(6)(4) + (9)(10)] + j[(4)(9) - (6)(10)]}{4^2 + 10^2}$

$= \frac{114 - j24}{116} = \mathbf{0.98 - j0.21}$

b. $\frac{(50 \angle 30°)(5 + j5)}{10 \angle -20°} = \frac{(50 \angle 30°)(7.07 \angle 45°)}{10 \angle -20°} = \frac{353.5 \angle 75°}{10 \angle -20°}$

$= 35.35 \underline{/75° - (-20°)} = \mathbf{35.35 \angle 95°}$

c. $\frac{(2 \angle 20°)^2(3 + j4)}{8 - j6} = \frac{(2 \angle 20°)(2 \angle 20°)(5 \angle 53.13°)}{10 \angle -36.87°}$

$= \frac{(4 \angle 40°)(5 \angle 53.13°)}{10 \angle -36.87°} = \frac{20 \angle 93.13°}{10 \angle -36.87°}$

$= 2 \underline{/93.13° - (-36.87°)} = \mathbf{2.0 \angle 130°}$

d. $3 \angle 27° - 6 \angle -40° = (2.673 + j1.362) - (4.596 - j3.857)$

$= (2.673 - 4.596) + j(1.362 + 3.857)$

$= \mathbf{-1.92 + j5.22}$