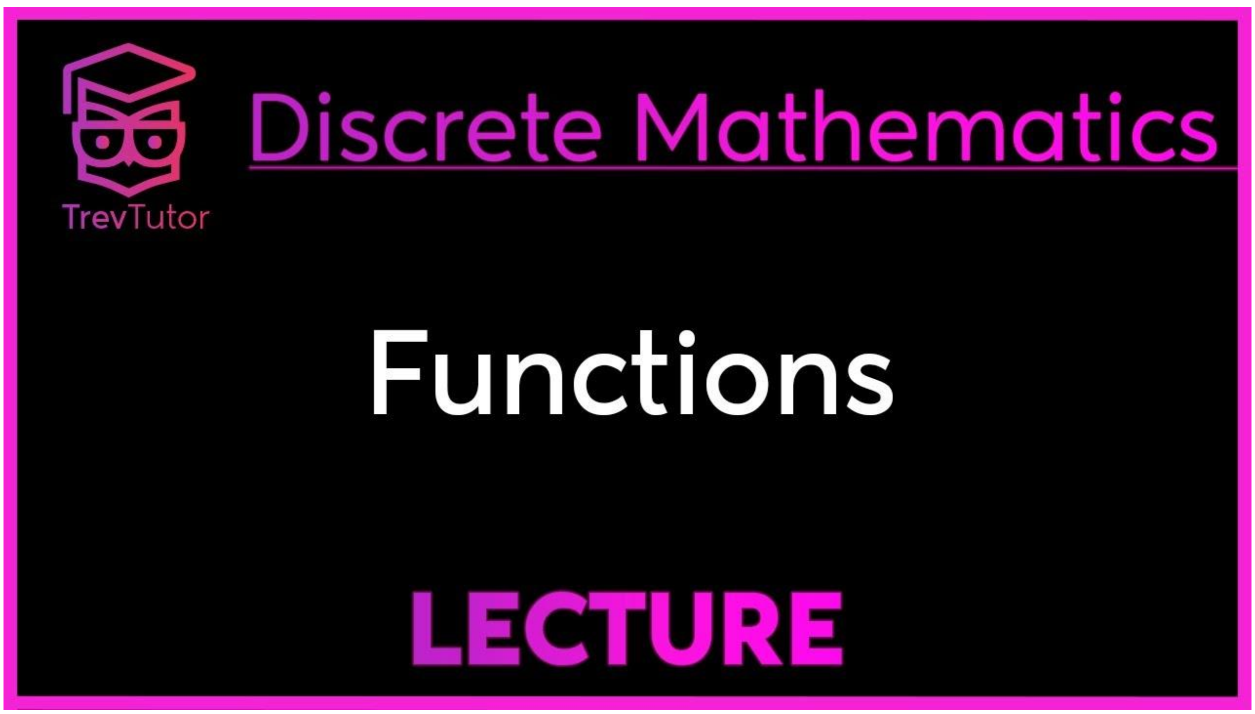
Chapter 3

هياكل متقطعة

المرحلة الاولى



Prepared by

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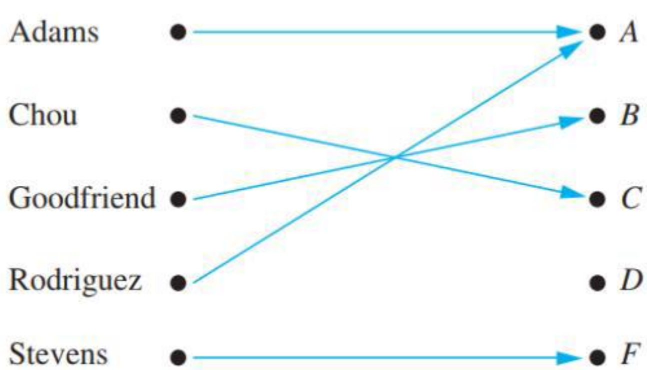
Function

Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A.

We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

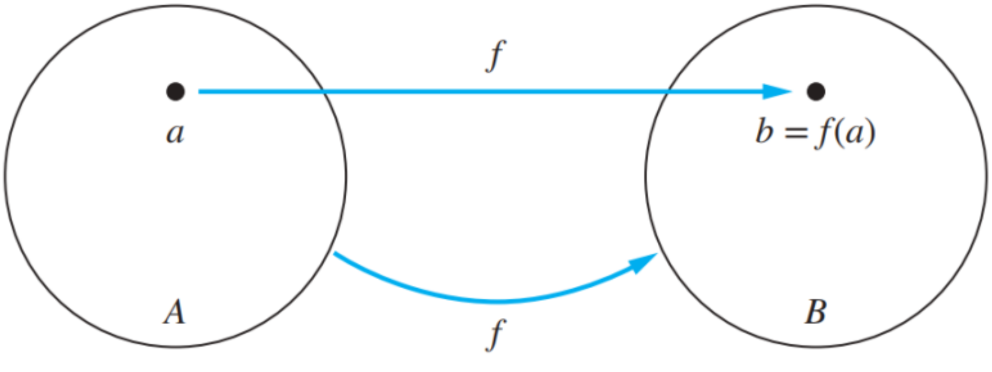
If f is a function from A to B, we write $f: A \to B$.

Function



Assignment of grades in a discrete mathematics class.

The Function $f: A \to B$



The function f maps A to B.

The Function $f: A \to B$

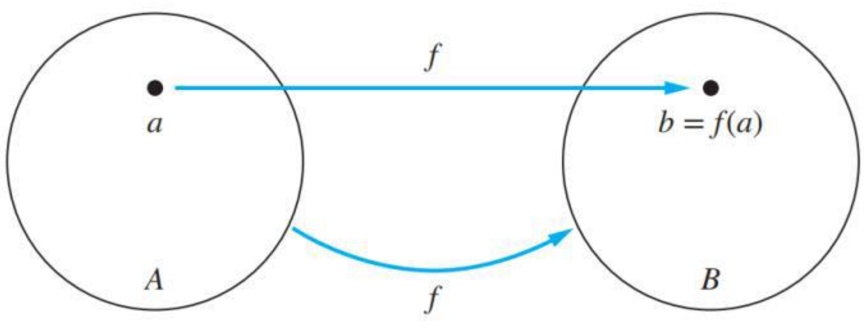
Domain: A

Co-Domain: B

$f(a) = b$

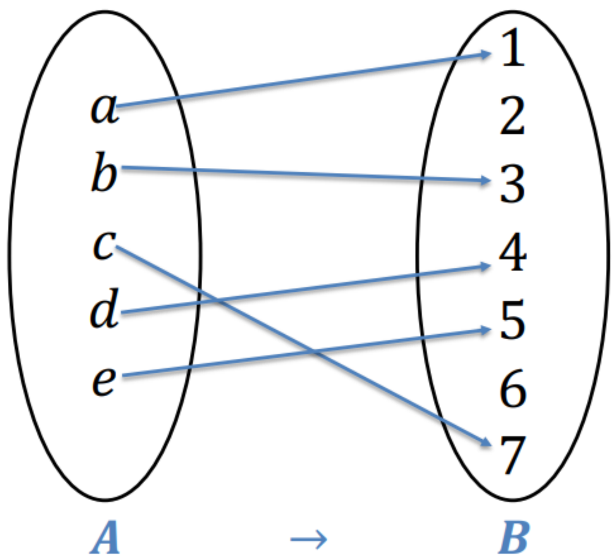
b is the ***image*** of a
a is a ***preimage*** of b

The **range**, or image, of f is the ***set of all images*** of elements of A.



The function f maps A to B.

The Function $f: A \to B$



Domain = $\{a, b, c, d, e\}$

Co-Domain = $\{1,2,3,4,5,6,7\}$

Range = $\{1,3,4,5,7\}$

Definition

Let f_1 and f_2 be functions from A to $\mathbf{R}$. Then $f_1 + f_2$ and $f_1\, f_2$ are also functions from A to $\mathbf{R}$ defined for all $x \in A$ by

$(f_1 + f_2)(x) = f_1(x) + f_2(x)$,

$(f_1 f_2)(x) = f_1(x) f_2(x)$.

Example

Let f_1 and f_2 be functions from **R** to **R** such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x,$

$(f_1 f_2)(x) = f_1(x) f_2(x) = x^2(x - x^2) = x^3 - x^4.$

Definition

Let f be a function from A to B and let S be a subset of A.

The image of S under the function f is the subset of B that consists of the images of the elements of S.

We denote the image of S by $f(S)$, so

$$f(S) = \{t \mid \exists s \in S\,(t = f(s))\}.$$

or shortly $\{f(s) \mid s \in S\}$.

Example

Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$, and $f(e) = 1$.

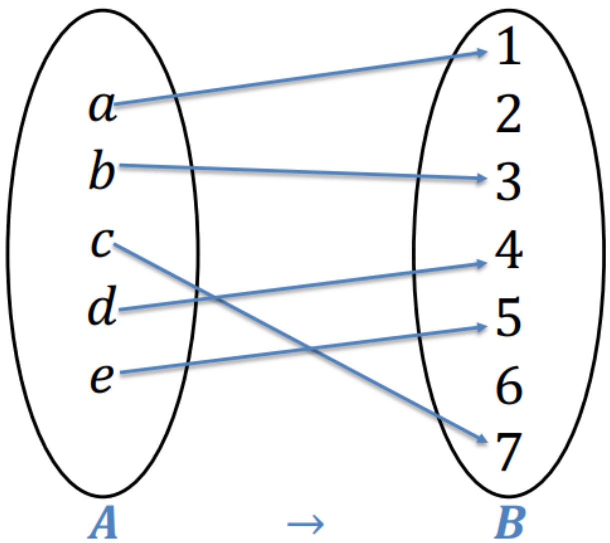
$S = \{b, c, d\} \subseteq A$

The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$

One-to-One function (injective)

A function f is said to be **one-to-one**, or **injective**,
if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.

One-to-One function (injective)



$f(a) = 1$

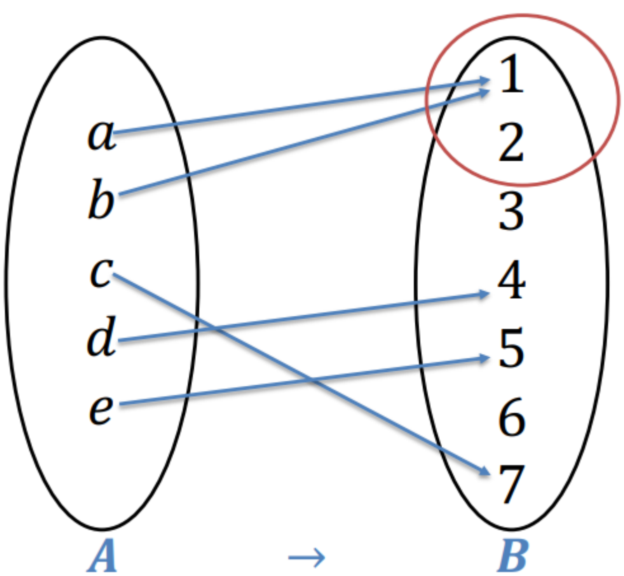
$f(b) = 3$

$f(c) = 7$

$f(d) = 4$

$f(e) = 5$

NOT *One-to-One* function (Not injective)



$f(a) = 1$

$f(b) = 1$

$f(c) = 7$

$f(d) = 4$

$f(e) = 5$

Solved Example on Injective Function

Example 1: Let's take a simple function, $f(x) = 2x$. Is this function injective?
Solution:
Yes, $f(x) = 2x$ is indeed an injective function. For every distinct input, you will always get a distinct output.

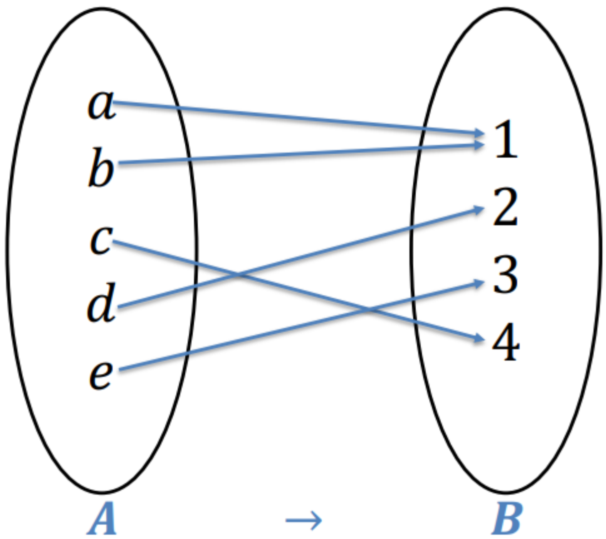
Example 2: Consider the function $f:R \to R$ defined as $f(x) = x^2$. Is this function injective?
Solution:
No, $f(x) = x^2$ is not an injective function because different inputs (e.g., $x = 2$ and $x = -2$) can result in the same output ($f(2) = 4$ and $f(-2) = 4$).

Example 3: Consider the function $f:R \to R$ defined as $f(x) = x^3$. Is this function injective?
Solution:
Yes, $f(x) = x^3$ is an injective function. Every unique input will result in a unique output.

onto function (surjective)

A function f from A to B is called **onto**, or **surjective**, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

onto function (surjective)



$f(a) = 1$

$f(b) = 1$

$f(c) = 4$

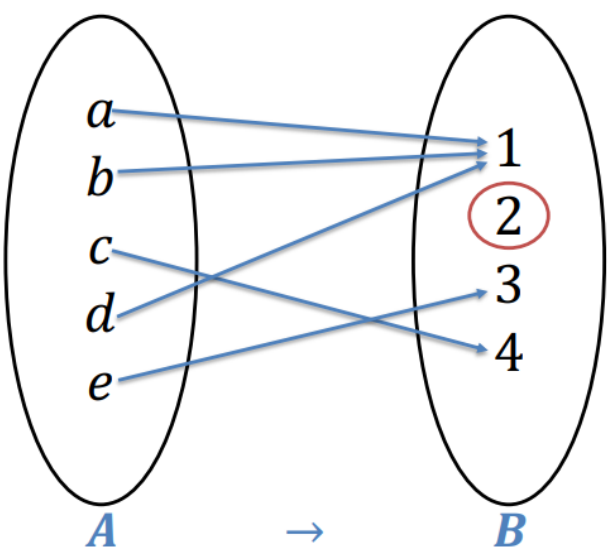
$f(d) = 2$

$f(e) = 3$

Co-Domain = {1,2,3,4}

Range = {1,2,3,4}

NOT *onto* function (Not surjective)



$f(a) = 1$

$f(b) = 1$

$f(c) = 4$

$f(d) = 1$

$f(e) = 3$

Co-Domain = {1,2,3,4}

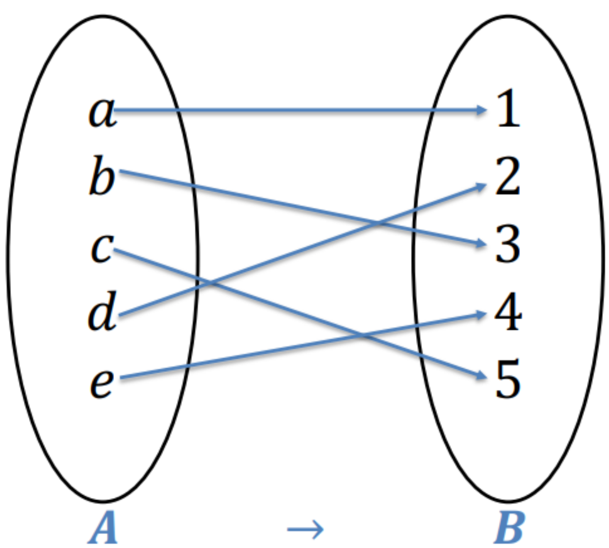
Range = {1,3,4}

One-to-one correspondence (bijection)

The function *f* is a **one-to-one correspondence**, or a **bijection**, if it is both one-to-one and onto.

One-to-one correspondence (bijection)

$|A| = |B|$



$f(a) = 1$

$f(b) = 3$

$f(c) = 5$

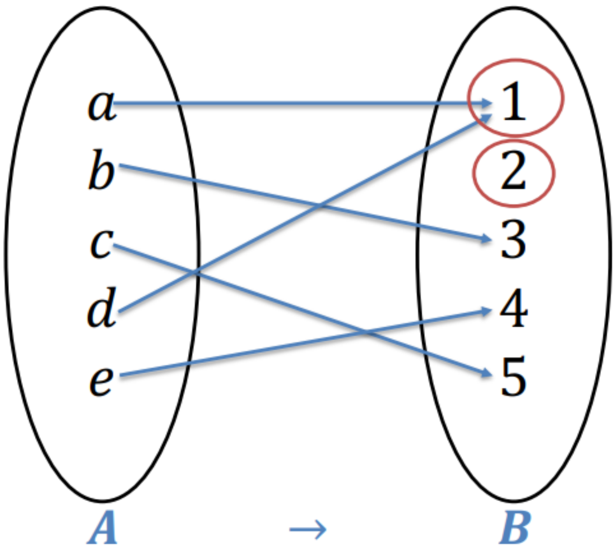
$f(d) = 2$

$f(e) = 4$

Co-Domain = {1,2,3,4,5}

Range = {1,2,3,4,5}

NOT *One-to-one correspondence* (Not bijection)



$f(a) = 1$

$f(b) = 3$ **NOT one-to-one**

$f(c) = 5$ **NOT onto**

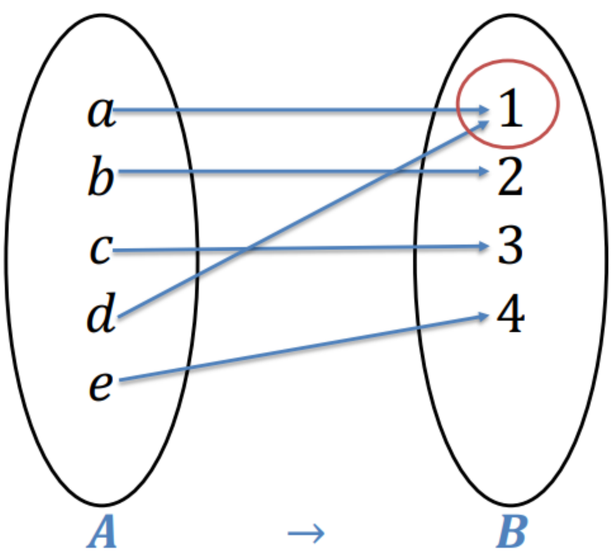
$f(d) = 1$

$f(e) = 4$

Co-Domain = {1,2,3,4,5}

Range = {1,3,4,5}

NOT *One-to-one correspondence* (Not bijection)



$f(a) = 1$

$f(b) = 2$ **Onto**

$f(c) = 3$ **NOT one-to-one**

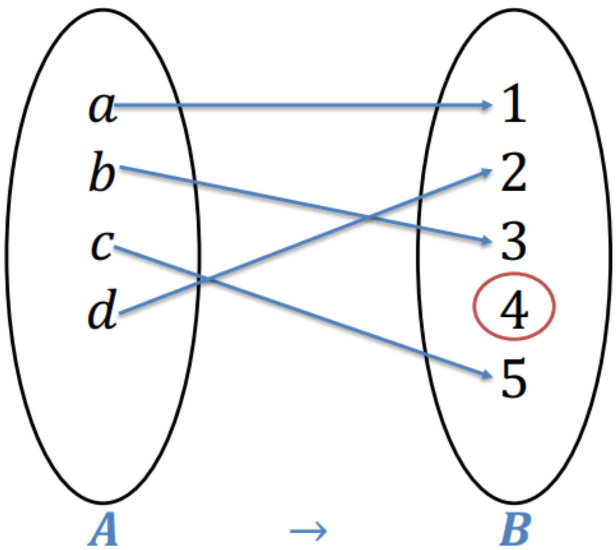
$f(d) = 1$

$f(e) = 4$

Co-Domain = {1,2,3,4}

Range = {1,2,3,4}

NOT *One-to-one correspondence* (Not bijection)



$f(a) = 1$

$f(b) = 3$ **One-to-one**

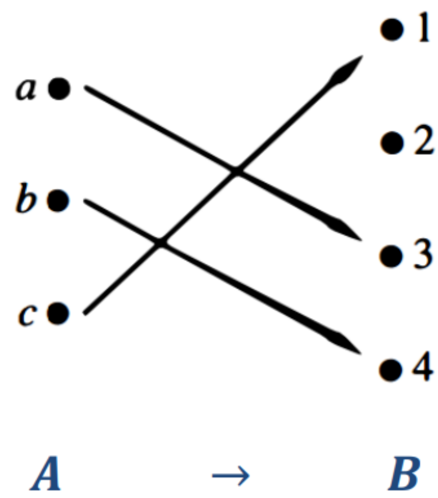
$f(c) = 5$ **NOT onto**

$f(d) = 2$

Co-Domain = {1,2,3,4,5}

Range = {1,2,3,5}

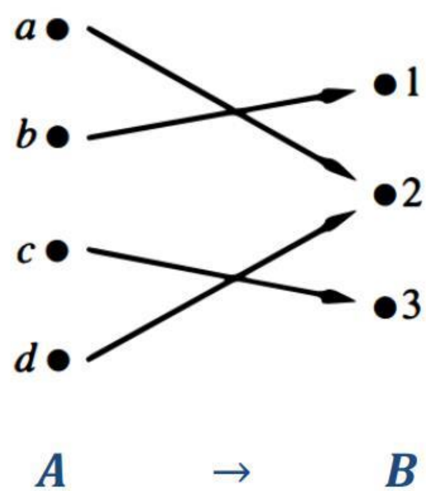
Examples



One-to-one

NOT onto

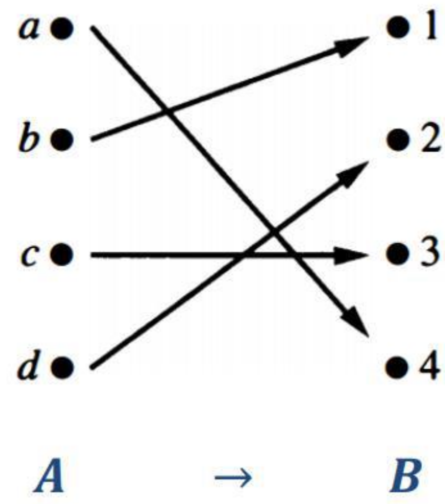
Examples



NOT One-to-one

Onto

Examples

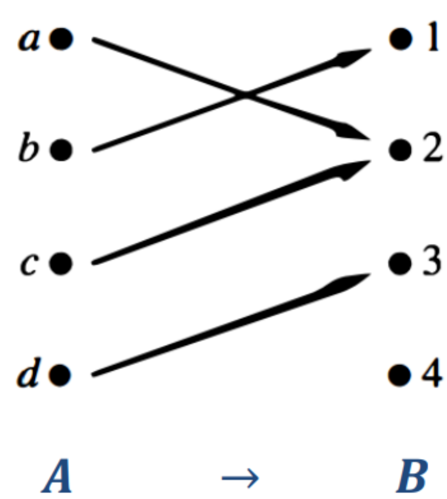


One-to-one

Onto

∴ bijection

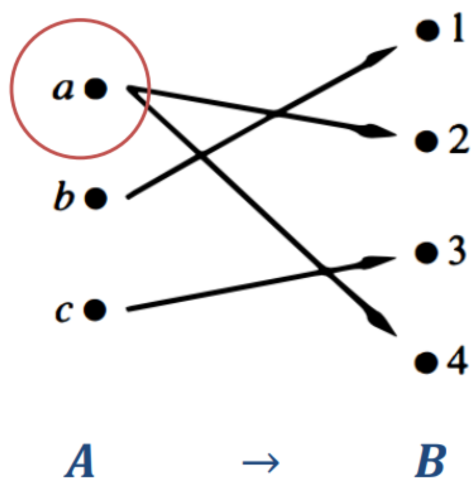
Examples



NOT One-to-one

NOT Onto

Examples



NOT a function from *A* to *B*

Examples (Answer)

Determine whether the function $f(x) = x + 1$ from the set of integers to the set of integers is one-to-one.

$f(a) = a + 1$ and $f(b) = b + 1$

$f(x)$ is one–to–one (if $f(a) = f(b)$ and a equal b then).

$$a + 1 = b + 1$$

$$a = b$$

$$\therefore f(x) \text{ is one–to–one}$$

Examples (Answer)

Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

$f(a) = a^2$ and $f(b) = b^2$

$f(x)$ is one–to–one (if $f(a) = f(b)$ and a equal b then).

$$a^2 = b^2$$

$$\pm a = \pm b$$

$$a \text{ may be not equal } b$$

$$\therefore f(x) \text{ is NOT one–to–one}$$

Solved Examples of Bijective Functions

Example 1: f(x) = x (Number to Itself)?

Solution:

Let's consider the set of natural numbers (positive whole numbers) as both Set A and Set B. In this case, you can define a bijective function like this:

f(1) = 1

f(2) = 2

f(3) = 3

...

In this function, every number in Set A (natural numbers) maps to the exact same number in Set B. It's a straightforward example of a bijective function because it meets both criteria each element in Set A matches a distinct element in Set B (injectivity) and no elements in Set B are left unmatched (surjectivity).

Example 3: Consider the function f:R→R defined as $f(x) = 2x+1$. Is this function bijective?

Solution:

- ***Injectivity:*** *To check injectivity, assume that $f(x_1) = f(x_2)$. for two different real numbers x_1 and x_2. This leads to the equation $2x_1 + 1 = 2x_2 + 1$. After some analysis, we find that x_1 must be equal to x_2. This proves that the function is injective.*
- ***Surjectivity:*** *To check surjectivity, we need to ensure that for any real number y, we can find a corresponding value of x such that $2x + 1 = y$. Solving this equation, we get $x = (y-1)/2$. This shows that the function covers all real numbers in its range, confirming its surjectivity.*

With both injectivity and surjectivity confirmed, the function f is indeed bijective.

Example 4: Let g: {1, 2, 3}→{a, b, c} be defined as $g(1) = a$, $g(2) = b$ and $g(3) = c$. Is g a bijective function?

Solution:

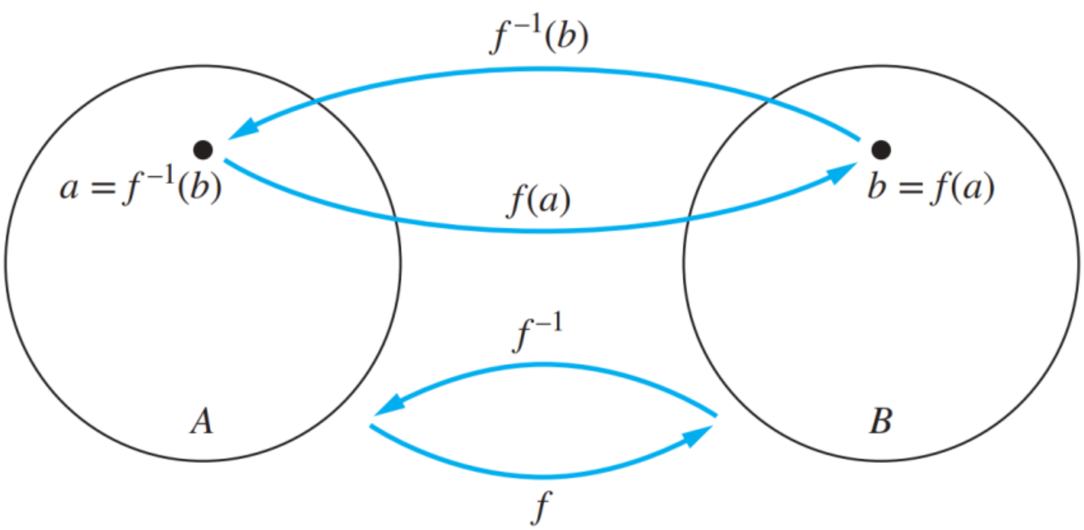
- **Injectivity:** In this case, each element from the domain (1, 2 and 3) maps to a distinct element in the codomain (a, b and c). This means the function is injective.
- **Surjectivity:** The function covers all the elements in the codomain (a, b and c). Since every element in the codomain has a connection in the domain, the function is surjective.

Considering these factors, g unquestionably qualifies as a **bijective** function.

Inverse Functions

Let f be a *one-to-one correspondence* from the set A to the set B. The **inverse** function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by $\boldsymbol{f^{-1}}$. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

Inverse Functions



Invertible

A one-to-one correspondence is called **invertible** because we can define an inverse of this function. A function is **not invertible** if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

Invertible – Example

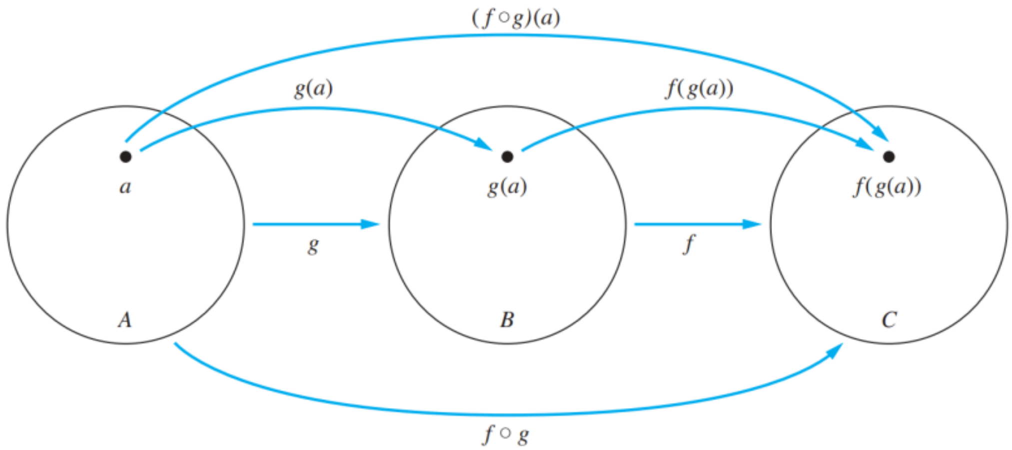
Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

Answer:

The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Composition of the Functions f and g

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$.



Note that the composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f.

Composition Example 1

Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of f and g, and what is the composition of g and f?

Answer:

1) The composition of f and g (i.e., $(f \circ g)$)

$(f \circ g)(a) = 2, \quad (f \circ g)(b) = 1, \quad (f \circ g)(c) = 3$

2) The composition of g and f (i.e., $(g \circ f)$) **cannot be defined** because the range of f is NOT a subset of the domain of g .

Composition Example 2

Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f ?

Answer:

1) The composition of f and g (i.e., $(f \circ g)$)

$(f \circ g)(x) = f\big(g(x)\big) = 2(3x + 2) + 3 = 6x + 7$

2) The composition of g and f (i.e., $(g \circ f)$)

$(g \circ f)(x) = g\big(f(x)\big) = 3(2x + 3) + 2 = 6x + 11$

Home works

[1] Let f_1 and f_2 be functions from $\mathbb{R}$ to $\mathbb{R}$ such that: $f_1(x) = x^2$, and $f_2(x) = x - x^2$. What is the function: $f_1 + f_2$?	
A.	x
B.	$2x^2$
C.	$x - 1$
D.	$2x^2 + x$

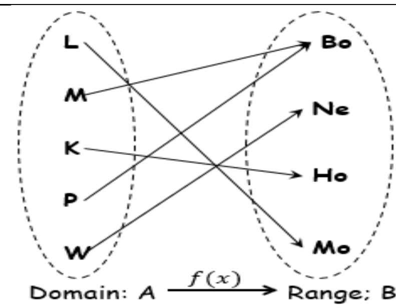
[2] Let f_1 and f_2 be functions from $\mathbb{R}$ to $\mathbb{R}$ such that: $f_1(x) = x^2$, and $f_2(x) = x - x^2$. What is the function: $f_1 f_2$?	
A.	$x^2(x^2 - 1)$
B.	$-x^4 - x^3$
C.	$x^2(x^2 + 1)$
D.	$-x^4 + x^3$

[3] A function $f: A \to B$ is said to be ……………………………………, if and only if $f(a) = f(b)$ implies that: $a = b$ for all a and b in the domain of f.	
A.	onto, or surjective
B.	one-to-one correspondence function.
C.	one-to-one (or injective)
D.	None of the above

[4] A function f: A→B is called ……………………………, if and only if for every element b∈B there is an element a∈A with f(a) = b.	
A.	onto, or surjective
B.	one-to-one correspondence function.
C.	one-to-one (or injective)
D.	None of the above

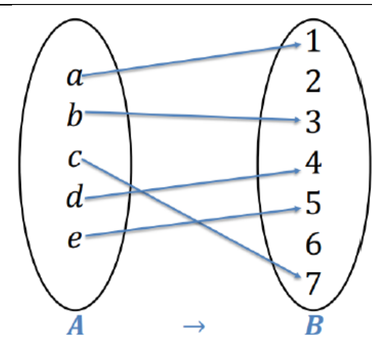
[5] A function f: A→B is a ………………………………, or a bijection, if and only if it is both one-to-one and onto.	
A.	onto function.
B.	one-to-one (or injective)
C.	onto, or surjective
D.	one-to-one correspondence function.

[6] Let the function: $f(x): A \to B$, the complete sets A and B are shown. A function $f(x)$ is:



A.	injective
B.	surjective
C.	bijective
D.	None of the above.

[7] Let the function: $f(x): A \to B$, the complete sets A and B are shown. A function $f(x)$ is:



A.	onto function.
B.	one-to-one correspondence function.
C.	one-to-one function.
D.	None of the above.

[8] Let $f(x)$ be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that: $f(a)=2, f(b)=3, and\ f(c)=1$. Is $f(x)$ invertible? And if it is, what is its inverse?	
A.	The function $f(x)$ is not invertible.
B.	The function $f(x)$ is invertible. And $f^{-1}(2)=a, f^{-1}(3)=b, and\ f^{-1}(1)=c$.
C.	The function $f(x)$ is invertible. And $f^{-1}(a)=2, f^{-1}(b)=3, and\ f^{-1}(c)=1$.
D.	None of the above.

[9] Let $f(x)\ and\ g(x)$ be the functions from the set of integers to the set of integers defined by: $f(x)=2x+3$, and $g(x)=3x+2$. What is the composition of $f(x)\ and\ g(x)$.	
A.	$f(g(x))=7x+6$
B.	$f(g(x))=6x+7$
C.	$f(g(x))=x^2$
D.	None of the above.

[10] Let $f(x)\ and\ g(x)$ be the functions from the set of integers to the set of integers defined by: $f(x)=2x+3$, and $g(x)=3x+2$. What is the composition of $g(x)\ and\ f(x)$.	
A.	$g(f(x))=6x+6$
B.	$g(f(x))=11x+6$
C.	$g(f(x))=6x+11$
D.	None of the above.

Practice Problems on Injective Function

Problem 1: Determine whether the following function is injective:

- f(x) = 2x + 3
- g(x) = x2 - 4x + 4
- k(x) = e^x
- q(x) = $x^3 + 2x^2 - x$
- u(x) = 3x - 2

Problem 2: Consider the function p(x) = 1/x for $x \neq 0$: Is p(x) an injective function?

Problem 3: Given the function r(x) = |x|, where x is a real number, is r(x) an injective function?

Problem 4: Consider the function $s(x) = \sqrt{x}$ for $x \geq 0$: Is s(x) injective?

Practice Problems on Onto Functions

Problem 1: Determine whether the function f: R → R defined by f(x) = 2x - 3 is onto.

Problem 2: Let g: Z → Z be defined by g(n) = 3n. Is the function g onto?

Problem 3: Consider the function h: {1, 2, 3, 4} → {5, 6, 7, 8} defined by h (1) = 5, h(2) = 6, h(3) = 7, and h(4) = 8. Is the function h onto?

Problem 4: Determine whether the function k: R → R defined by $k(x) = x^2$ is onto.

Problem 5: Let m: N → N be defined by m(n) = n + 1. Is the function m onto?

Problem 6: Consider the function p: {a, b, c, d, e} → {1, 2, 3} defined by p(a) = 1, p(b) = 1, p(c) = 2, p(d) = 2, and p(e) = 3. Is the function p onto?

Practice Problems on Bijective Function

Problem 1: Determine whether the following function is Bijective:

- f(x) = 2x + 5
- $g(x) = x^2 + 1$
- k(x) = 5x – 2

Problem 2: Consider the function $f(x) = 1/(x - 10)$ for $x \neq 10$: Is $p(x)$ an bijective function?

Question 3: Given two functions $f(x) = x + 2$ and $h(x) = 2x – 3$

- Find the composite function $(f \circ h)(x)$
- Evaluate $(f \circ h)(2)$