CHAPTER 2

Limits and Continuity

2.1 Limits

If the values of $f(x)$ tend to get closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$.

I DEFINITION We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

Example 1: Find the limit of the function $f(x) = x^2 - x + 2$ as x approaches 2.

Solution:

Let's investigate the behavior of the function $f(x) = x^2 - x + 2$ for values of x near 2. The following table gives values of $f(x)$ for values of x close to 2, but not equal to 2.

x	f(x)	x	f(x)
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

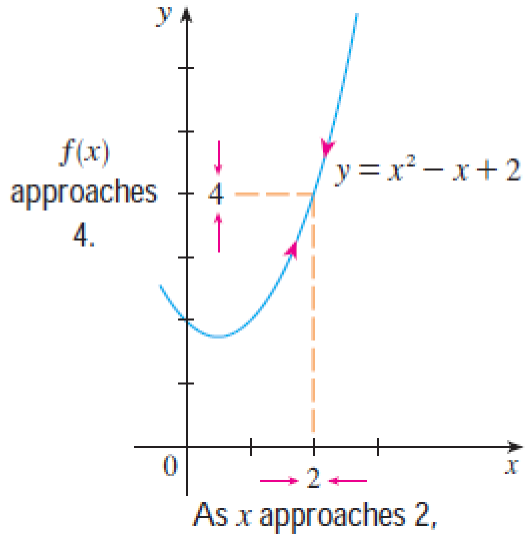


Figure 1

From the table and the graph of *f* (a parabola) shown in Figure 1 we see that when *x* is close to 2 (on either side of 2), *f (x)* is close to 4. In fact, it appears that we can make the values of *f (x)* as close as we like to 4 by taking *x* sufficiently close to 2. We express this by saying "the limit of the function $f(x) = x^2 - x + 2$ as *x* approaches 2 is equal to 4."
The notation for this is:

$$\lim_{x \to 2}(x^2 - x + 2) = 4$$

Example 2: Guess the value of $\lim_{x \to 1} \frac{x-1}{x^2-1}$
Solution:
Notice that the function $f(x) = (x - 1) / (x^2 - 1)$ is not defined when $x = 1$, but that doesn't matter because the definition of $\lim_{x \to a} f(x)$ says that we consider values of *x* that are close to *a* but not equal to *a*.
The tables below give values of *f (x)* (correct to six decimal places) for values of *x* that approach 1 (but are not equal to 1).

x < 1	f (x)
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

x > 1	f (x)
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

On the basis of the values in the tables, we make the guess that

$$\lim_{x\to 1}\frac{x-1}{x^2-1}=0.5$$

The value of $\lim_{x\to 1}\frac{x-1}{x^2-1}$ can be solved to give same results by:

$$\lim_{x\to 1}\frac{x-1}{x^2-1}=\lim_{x\to 1}\frac{(x-1)}{(x-1)(x+1)}$$

$$\lim_{x\to 1}\frac{1}{x+1}$$

$$\lim_{x\to 1}\frac{1}{(1)+1}=0.5$$

Example 2 is illustrated by the graph of in Figure 2.

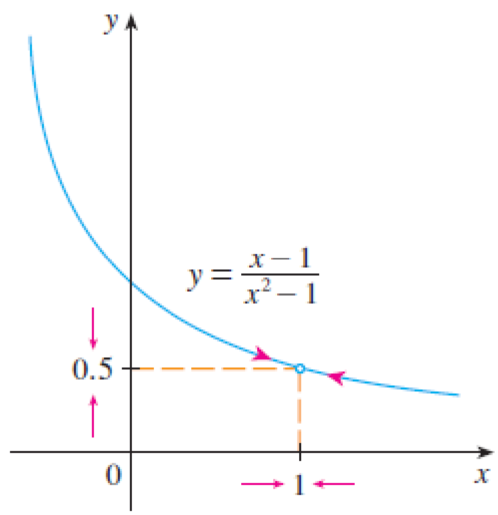


Figure 2

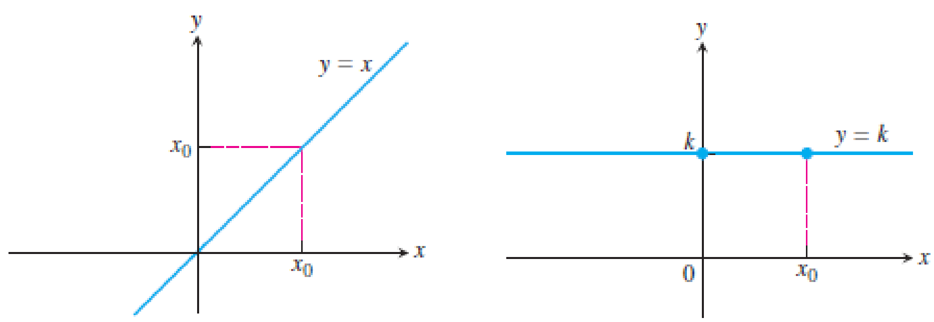
Example: 3

(a) If f is the **identity function** $f(x) = x$**,** then for any value of x_0 (Figure 3a),

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0$$

(b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of x_0 (Figure 3b),

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k$$



(a) Identity function (b) Constant function

Figure 3

Example 4: Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.

Solution: As x becomes close to 0, x^2 also becomes close to 0, and $1/x^2$ becomes very large. (See the table).

x	$\frac{1}{x^2}$
±1	1
±0.5	4
±0.2	25
±0.1	100
±0.05	400
±0.01	10,000
±0.001	1,000,000

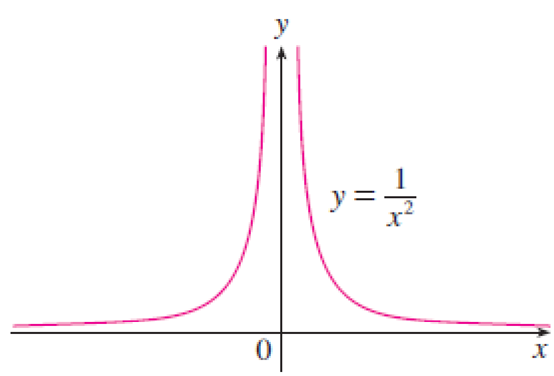


Figure 4

In fact, it appears from the graph of the function $f(x) = 1/x^2$ shown in Figure 4 that the values of $f(x)$ can be made arbitrarily large by taking x close enough to 0. Thus the values of $f(x)$ do not approach a number, so $\lim_{x\to 0} (1/x^2)$ **does not exist**.
To indicate the kind of behavior exhibited in Example 4, we use the notation:

$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

Example 5: Investigate $\lim_{x\to 0} \sin \frac{\pi}{x}$

Solution:

Again the function $f(x) = \sin(\pi/x)$ is undefined at 0. The graph of function was shown in Figure 5.

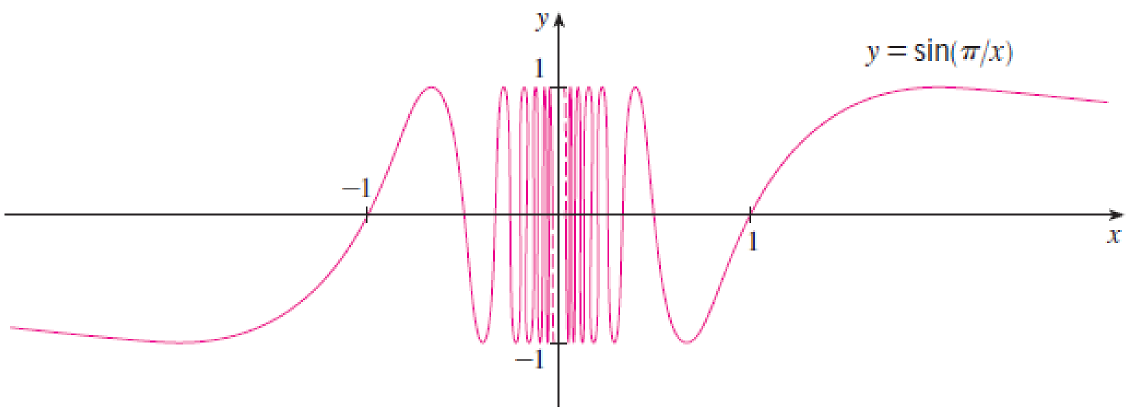


Figure 5

The graph indicate that the values of $\sin(\pi/x)$ oscillate between 1 and -1 infinitely often as x approaches 0.
Since the values of do not approach a fixed number as approaches 0,

$$\lim_{x\to 0} \sin \frac{\pi}{x} \quad \text{do not exist}$$

2.1.1 The Limit Laws

To calculate limits of functions that are arithmetic combinations of functions having known limits, we can use several easy rules:

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$.)

Example 6: Use the observations $\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$ (Example 3) and the properties of limits to find the following limits:

(a) $\lim_{x\to c}(x^3 + 4x^2 - 3)$ (b) $\lim_{x\to c} \frac{x^4+x^2-1}{x^2+5}$ (c) $\lim_{x\to -2} \sqrt{4x^2 - 3}$

Solution:

(a) $\lim_{x\to c}(x^3 + 4x^2 - 3) = \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3$ Sum and Difference Rules

$= c^3 + 4c^2 - 3$ Power and Multiple Rules

(b) $$\lim_{x\to c}\frac{x^4+x^2-1}{x^2+5} = \frac{\lim_{x\to c}(x^4+x^2-1)}{\lim_{x\to c}(x^2+5)}$$ Quotient Rule

$$= \frac{\lim_{x\to c}x^4 + \lim_{x\to c}x^2 - \lim_{x\to c}1}{\lim_{x\to c}x^2 + \lim_{x\to c}5}$$ Sum and Difference Rules

$$= \frac{c^4+c^2-1}{c^2+5}$$ Power or Product Rule

(c) $$\lim_{x\to -2}\sqrt{4x^2-3} = \sqrt{\lim_{x\to -2}(4x^2-3)}$$ Root Rule with $n = 2$

$$= \sqrt{\lim_{x\to -2}4x^2 - \lim_{x\to -2}3}$$ Difference Rule

$$= \sqrt{4(-2)^2-3}$$ Product and Multiple Rules

$$= \sqrt{16-3}$$

$$= \sqrt{13}$$

THEOREM 2—Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x\to c}\frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example 7: The following calculation illustrates Theorems 2 and 3:

$$\lim_{x\to -1}\frac{x^3+4x^2-3}{x^2+5} = \frac{(-1)^3+4(-1)^2-3}{(-1)^2+5} = \frac{0}{6} = 0$$

Example 8: Evaluate $\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}$

Solution: We can create a common factor by multiplying both numerator and denominator by the conjugate radical expression $\sqrt{x^2+100}+10$ (obtained by changing the sign after the square root). The preliminary algebra rationalizes the numerator:

$$\frac{\sqrt{x^2+100}-10}{x^2} = \frac{\sqrt{x^2+100}-10}{x^2}\cdot\frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$$

$$= \frac{x^2+100-100}{x^2\left(\sqrt{x^2+100}+10\right)}$$

$$= \frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)} \qquad \text{Common factor } x^2$$

$$= \frac{1}{\sqrt{x^2+100}+10}. \qquad \text{Cancel } x^2 \text{ for } x \neq 0$$

Therefore,

$$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x\to 0}\frac{1}{\sqrt{x^2+100}+10}$$

$$= \frac{1}{\sqrt{0^2+100}+10} \qquad \text{Denominator not 0 at } x = 0\text{; substitute}$$

$$= \frac{1}{20} = 0.05.$$

2.1.2 Indeterminate Forms:

There are seven indeterminate forms:

$0/0$, ∞/∞, $0.\infty$, $\infty-\infty$, 0^0, ∞^0, and 1^∞

2.1.3 Sandwich Theorem

The following theorem enables us to calculate a variety of limits. It is called the Sandwich Theorem because it refers to a function f whose values are sandwiched between the values of two other functions g and h that have the same limit L at a point c. See Figure 6

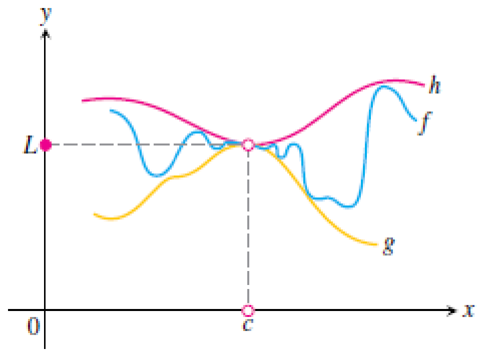


Figure 6

THEOREM 4—The Sandwich Theorem Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

The Sandwich Theorem is also called the Squeeze Theorem or the Pinching Theorem.

Example 9: Given that

$$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2} \quad \text{for all } x \neq 0$$

find $\lim_{x \to 0} u(x)$, no matter how complicated u is.
Solution: Since

$$\lim_{x \to 0} (1 - (x^2/4)) = 1 \quad \text{and} \lim_{x \to 0} (1 + (x^2/2)) = 1,$$

the Sandwich Theorem implies that $\lim_{x \to 0} u(x) = 1$ (Figure 7).

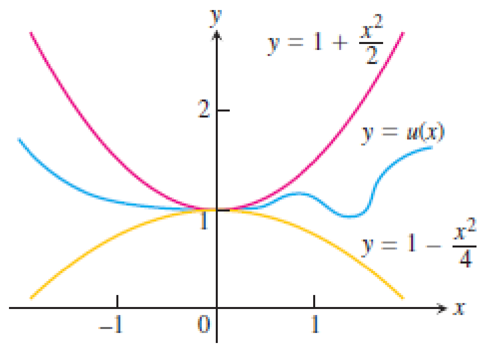


Figure 7

Example 10: Show that $\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$.
Solution: First note that we **cannot** use

$$\lim_{x \to 0} x^2 \sin\frac{1}{x} = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin\frac{1}{x}$$

because $\lim_{x \to 0} \sin(1/x)$ does not exist (see Example 4 in Section 2.2). However, since

$$-1 \leqslant \sin\frac{1}{x} \leqslant 1$$

we have, as illustrated by Figure 8,

$$-x^2 \leqslant x^2 \sin\frac{1}{x} \leqslant x^2$$

We know that

$$\lim_{x \to 0} x^2 = 0 \qquad \text{and} \qquad \lim_{x \to 0} (-x^2) = 0$$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$$

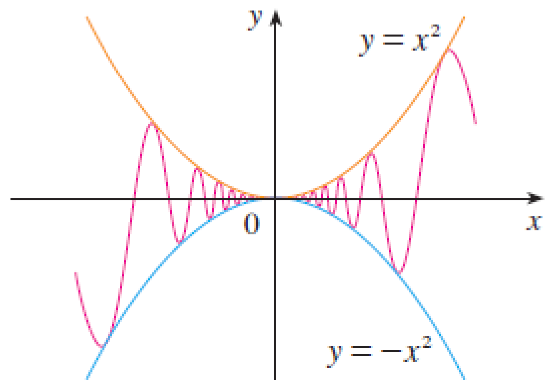


Figure 8

2.1.4 One-Sided Limits

In this section we extend the limit concept to *one-sided limits*, which are limits as x approaches the number c from the left-hand side (where $x < c$) or the right-hand side ($x > c$) only.

THEOREM 6 A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

In another word:

- Right-hand limit is the limit of *f(x)* as x approaches c from the right, or $\lim_{x \to c^+} f(x)$
- Left-hand limit is the limit of *f(x)* as x approaches c from the left, or $\lim_{x \to c^-} f(x)$
- $\lim_{x \to c} f(x)$ = L if and only if $\lim_{x \to c^+} f(x) = L$ and $\lim_{x \to c^-} f(x) = L$

Example 11: Let

$$f(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{x}{2} + 1, & x > 2 \end{cases}$$

(a) Find $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$
(b) Does $\lim_{x\to 2} f(x)$ exist ? why ?

Solution:

(a) $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} f(\frac{x}{2} + 1)$ = 2/2 + 1 = 2

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(3 - x)$ = 3 – 2 = 1

(b) $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$

$\lim_{x\to 2} f(x)$ does not exist

Example 12: let $f(x) = \begin{cases} 1 - x^2, & x \neq 1 \\ 2, & x = 1 \end{cases}$

(a) Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$
(b) Does $\lim_{x\to 1} f(x)$ exist ? why ?
(c) Graph $f(x)$

Solution:

(a) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 1 - x^2 = 1 - (1)^2 = 0$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 1 - x^2 = 1 - (1)^2 = 0$

(b) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x)$, $\lim_{x\to 1} f(x)$ exist

2.1.5 Limits Involving (sin θ/θ)

A central fact about (sin θ/θ) is that in radian measure its limit as $\theta \to 0$ is 1.

THEOREM 7

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \qquad (1)$$

Proof The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well.

To show that the right-hand limit is 1, we begin with positive values of less than $\pi/2$ (Figure 9).

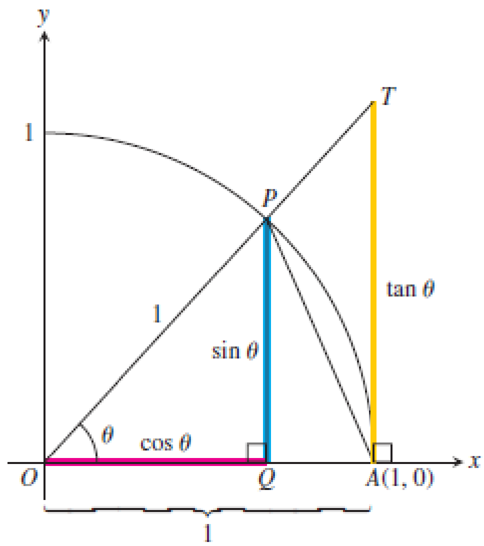


Figure 9

$$\text{Area } \Delta OAP < \text{ area sector } OAP < \text{ area } \Delta OAT.$$

We can express these areas in terms of θ as follows:

$$\text{Area } \Delta OAP = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\sin\theta) = \frac{1}{2}\sin\theta$$

$$\text{Area sector } OAP = \frac{1}{2}r^2\theta = \frac{1}{2}(1)^2\theta = \frac{\theta}{2} \qquad (2)$$

$$\text{Area } \Delta OAT = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta.$$

Thus,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

This last inequality goes the same way if we divide all three terms by the number $(1/2)\sin\theta$, which is positive since $0 < \theta < \pi/2$:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Taking reciprocals reverses the inequalities:

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim_{\theta \to 0^+} \cos\theta = 1$, the Sandwich theorem gives:

$$\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1.$$

Recall that $\sin\theta$ and θ are both *odd functions* (Section 1.1). Therefore, $f(\theta) = (\sin\theta)/\theta$ is an *even function*, with a graph symmetric about the y-axis (see Figure 2.32). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta \to 0^-} \frac{\sin\theta}{\theta} = 1 = \lim_{\theta \to 0^+} \frac{\sin\theta}{\theta},$$

so $\lim_{\theta \to 0} (\sin\theta)/\theta = 1$ by Theorem 6. ■

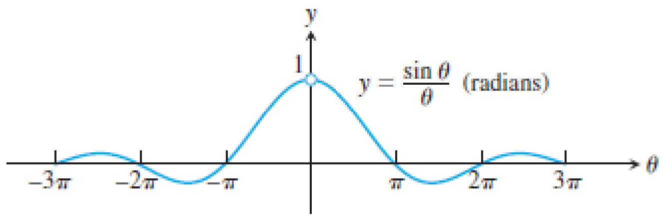


Figure 10

Example 13: show that

(a) $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$ and **(b)** $\lim_{x \to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$

Solution:

(a) Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\begin{aligned}\lim_{h \to 0} \frac{\cos h - 1}{h} &= \lim_{h \to 0} -\frac{2\sin^2(h/2)}{h} \\ &= -\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \sin\theta \qquad \text{Let } \theta = h/2. \\ &= -(1)(0) = 0. \qquad \text{Eq. (1) and Example 11a in Section 2.2}\end{aligned}$$

(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\lim_{x\to 0}\frac{\sin 2x}{5x} = \lim_{x\to 0}\frac{(2/5)\cdot \sin 2x}{(2/5)\cdot 5x}$$

$$= \frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x}$$

Now, Eq. (1) applies with $\theta = 2x$.

$$= \frac{2}{5}(1) = \frac{2}{5}$$

■

Example 14: Find $\lim_{t\to 0}\frac{\tan t \sec 2t}{3t}$

Solution From the definition of $\tan t$ and $\sec 2t$, we have

$$\lim_{t\to 0}\frac{\tan t \sec 2t}{3t} = \frac{1}{3}\lim_{t\to 0}\frac{\sin t}{t}\cdot\frac{1}{\cos t}\cdot\frac{1}{\cos 2t}$$

$$= \frac{1}{3}(1)(1)(1) = \frac{1}{3}.$$

Eq. (1) and Example 11b in Section 2.2

■

2.1.6 Limits at Infinity

General rules:

$\lim_{x\to\infty}\frac{1}{x} = 0,\qquad \lim_{x\to -\infty}\frac{1}{x} = 0$

$\lim_{x\to\infty} k = k,\qquad \lim_{x\to -\infty} k = k$

$\lim_{x\to\infty}\frac{\sin\theta}{\theta} = 0$, to prove it:

$-1 \le \sin\theta \le 1 \quad [\div\ \theta] \qquad \frac{-1}{\theta} \le \frac{\sin\theta}{\theta} \le \frac{1}{\theta}$

$\lim_{\theta\to\infty}\frac{-1}{\theta} = 0$ and $\lim_{\theta\to\infty}\frac{1}{\theta} = 0$, then from Sandwich theorem:

$$\lim_{\theta\to\infty}\frac{\sin\theta}{\theta} = 0$$

Limits at Infinity of Rational Functions

There are two methods:

1. Divide both the numerator and the denominator by the highest power of x in denominator.
2. Suppose that $x = 1/h$ and find limit as h approaches zero.

Note: for rational function $\frac{f(x)}{g(x)}$

1. If degree of $f(x)$ less than degree of $g(x)$, then $\lim_{x\to\pm\infty}\frac{f(x)}{g(x)} = 0$
2. If degree of $f(x)$ equals degree of $g(x)$, then $\lim_{x\to\pm\infty}\frac{f(x)}{g(x)}$ is finite.
3. If degree of $f(x)$ greater than degree of $g(x)$, then $\lim_{x\to\pm\infty}\frac{f(x)}{g(x)}$ is infinite.

Example 15: find $\lim_{x\to\infty}\frac{x^3-4x^2+7}{2x^2-3}$

Method 1: $\lim_{x\to\infty}\frac{\frac{x^3}{x^2}-\frac{4x^2}{x^2}+\frac{7}{x^2}}{\frac{2x^2}{x^2}-\frac{3}{x^2}} = \lim_{x\to\infty}\frac{x-4+\frac{7}{x^2}}{2-\frac{3}{x^2}} = \frac{\infty-4+0}{2-0} = \frac{\infty}{2} = \infty$

Method 2: let $x = 1/h$, $h \to 0$

$$\lim_{h\to0}\frac{(\frac{1}{h})^3-4\left(\frac{1}{h}\right)^2+7}{2(\frac{1}{h})^2-3} = \lim_{h\to0}\frac{\frac{1}{h^3}-\frac{4}{h^2}+7}{\frac{2}{h^2}-3} = \lim_{h\to0}\frac{\frac{1-4h+7h^3}{h^3}}{\frac{2-3h^2}{h^2}}$$

$$\lim_{h\to0}\frac{1-4h+7h^3}{h(2-3h^2)} = \frac{1-4(0)+7(0)}{0(2-3(0))} = \frac{1}{0} = \infty$$

Example 16: Find $\lim_{x\to\infty}\frac{x^{3/2}+5}{\sqrt{x^3+4}}$

$$\lim_{x\to\infty}\frac{\frac{x^{3/2}}{x^{3/2}}+\frac{5}{x^{3/2}}}{\sqrt{\frac{x^3}{x^3}+\frac{4}{x^3}}} = \lim_{x\to\infty}\frac{1+\frac{5}{x^{3/2}}}{\sqrt{1+\frac{4}{x^3}}} = \frac{1+0}{\sqrt{1+0}} = \frac{1}{\sqrt{1}} = 1$$

2.1.7 Absolute Value in Limit Problems

Example 17: Find $\lim_{x\to-2}(x+3)\frac{|x+2|}{x+2}$

Solution:

$\lim_{x\to-2^+}(x+3)\frac{|x+2|}{x+2} = \lim_{x\to-2^+}(x+3)\frac{(x+2)}{(x+2)}$

$= \lim_{x\to-2^+}(x+3)$ = - 2 + 3 = 1

$\lim_{x\to-2^-}(x+3)\frac{|x+2|}{x+2} = \lim_{x\to-2^-}(x+3)\frac{-(x+2)}{(x+2)}$

$= \lim_{x\to-2^-}(x+3)(-1)$ = (- 2 + 3) (-1) = -1

2.2 Continuity

We noticed that the limit of a function as *x* approaches *a* can often be found simply by calculating the value of the function at *a*. Functions with this property are called *continuous at a.*

1 DEFINITION A function f is **continuous at a number** a if

$$\lim_{x \to a} f(x) = f(a)$$

Notice that Definition l implicitly requires three things if *f* is continuous at *a*:

1. $f(a)$ is defined (that is, a is in the domain of f)

2. $\lim_{x\to a} f(x)$ exists

3. $\lim_{x\to a} f(x) = f(a)$

Example 18: Where are each of the following functions discontinuous?

(a) $f(x) = \frac{x^2-x-2}{x-2}$

(b) $f(x) = \begin{cases} \frac{1}{x^2} & if\ x \neq 0 \\ 1 & if\ x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & if\ x \neq 2 \\ 1 & if\ x = 2 \end{cases}$

Solution:

(a) Notice that $f(2)$ is not defined, so f is discontinuous at 2. Later we'll see why is continuous at all other numbers.

(b) Here $f(0)$ is defined but

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} 1/x^2$$

does not exist. So f is discontinuous at 0.

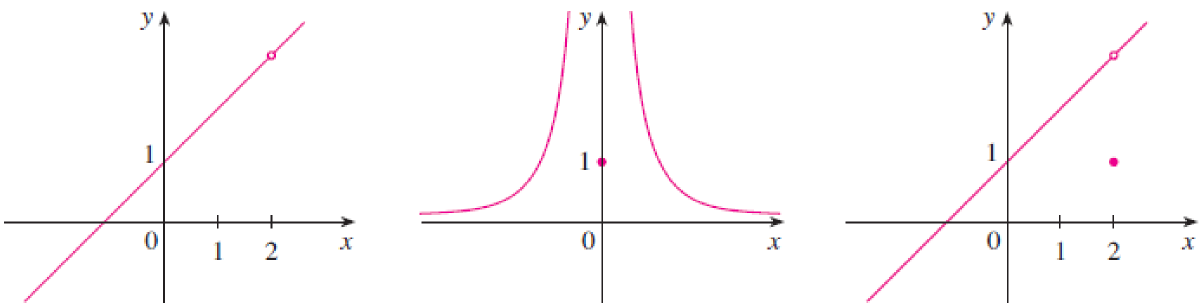
(c) Here $f(2)$ is defined and

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{x^2-x-2}{x-2} = \lim_{x\to 2}\frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2}(x+1) = 3$$

exists. But

$$\lim_{x\to 2} f(x) \neq f(2)$$

so f is not continuous at 2. See Figure 11



(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ (b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ (c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

Figure 11

2 DEFINITION A function f is **continuous from the right at a number** a if

$$\lim_{x\to a^+} f(x) = f(a)$$

and f is **continuous from the left at** a if

$$\lim_{x\to a^-} f(x) = f(a)$$

3 DEFINITION A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left.)

Example 19: check the continuity of $f(x)$ at $x = 0, 1, 2, 3$ and 4

$$\begin{cases} 2-x, & 0 \le x \le 1 \\ 2, & 1 \le x \le 2 \\ 3, & x = 2 \\ 2x-2, & 2 < x \le 3 \\ 10-2x, & 3 < x \le 4 \end{cases}$$

Solution:

At $x = 0$:

1. $f(0) = 2 - 0 = 0$
2. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2 - x = 2 - 0 = 2$
3. $f(0) = \lim_{x\to 0^+} f(x)$

the function is continuous at $x = 0$.

At $x = 1$:

1. $f(1) = 2$
2. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2 = 2$
   $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2 - x = 2 - 1 = 1$
3. $2 \neq 1 \quad \lim_{x\to 1} f(x)$ does not exist
   The function is not continuous at $x = 1$

At $x = 2$:

1. $f(2) = 3$
2. $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 2x - 2 = 2(2) - 2 = 2$
   $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2 = 2$
3. $f(2) \neq \lim_{x\to 2}$ the function is not continuous at x = 2

At $x = 3$:

1. $f(3) = 2(3) - 2 = 4$
2. $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 10 - 2x = 10 - 2(3) = 4$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 2x - 2 = 2(3) - 2 = 4$

$4 = 4 \quad \lim_{x\to 3} f(x) = 4$

3. $f(3) = \lim_{x\to 3} f(x)$ the function is continuous at $x = 3$

<u>At $x = 4$:</u>

1. $f(4) = 10 - 2(4) = 2$
2. $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} 10 - 2x = 10 - 2(4) = 2$
3. $f(4) = \lim_{x\to 4^-} f(x)$ the function is continuous at $x = 4$.

By graphing the function we can check the continuity of the function as shown in Figure

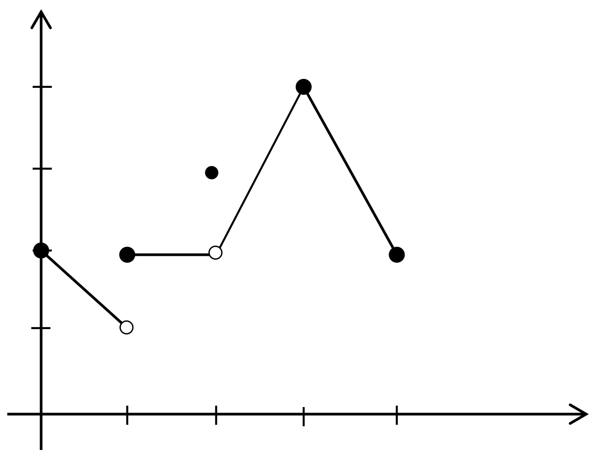


Figure 12

Example 20: At what points is the function f (x) continuous?

$$f(x) = \begin{cases} 1, & x < 0 \\ \sqrt{1 - x^2}, & 0 \le x \le 1 \\ x - 1, & x > 1 \end{cases}$$

Solution:

<u>At $x = 0$</u>

1. $f(0) = \sqrt{1 - (0)^2} = \sqrt{1} = 1$
2. $\lim_{x\to 0^+} f(x) = \sqrt{1 - (0)^2} = \sqrt{1} = 1$

   $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^1} 1 = 1$

$\lim_{x \to 0} f(x) = 1$

3. $f(0) = \lim_{x \to 0} f(x)$ *f (x)* is continuous at *x* = 0

<u>At *x* = 0</u>

1. $f(1) = 0$
2. $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x - 1 = 1 - 1 = 0$

   $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \sqrt{1 - (1)^2} = 0$

   $\lim_{x \to 1} f(x) = 0$
3. $f(1) = \lim_{x \to 1} f(x)$ *f (x)* continuous at *x* = 1

<u>For *x* < 0:</u>

f (*x*) is continuous function

<u>For *x* > 1:</u>

f (*x*) = *x* – 1 is continuous function

The function is continuous at every point

Example 21: Find the points at which $y = (x^3 - 1) / (x^2 - 1)$ is discontinuous.

Solution:

The function to be discontinuous, the denominator must equal to **zero**:

So that, $x^2 - 1 = 0$ $x^2 = 1$ $x = \pm 1$

Example 22: What value should be assigned to *a* to make the function

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at *x* = 3 ?

Solution: to make $f(x)$ continuous at $x = 3$:

$\lim_{x\to 3} f(x) = f(3)$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x^2 - 1 = (3)^2 - 1 = 8$

$\lim_{x\to 3} f(x) = 8$

$f(3) = 2ax = 2 * a * 3 = 6a$

$8 = 6a \quad a = 6/8 = 4/3$

3 DEFINITION A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left.)

Example 23: Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval [-1, 1].

Solution:
The function does not have a left-hand limit at $x = -1$ or a right-hand limit at $x = 1$.
$\lim_{x\to -1^+} f(x) = 1 = f(-1)$ and
$\lim_{x\to 1^-} f(x) = 1 = f(1)$
So f is continuous from the right at -1 and continuous from the left at 1. Therefore, according to Definition, f is continuous on [-1, 1].
The graph of is sketched in Figure . It is the lower half of the circle:

$$x^2 + (y - 1)^2 = 1$$

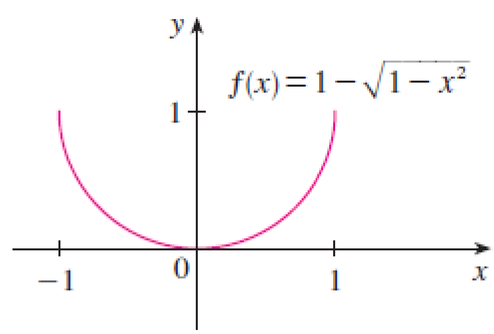


Figure 13

THEOREM 8—Properties of Continuous Functions If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Constant multiples:* $k \cdot f$, for any number k
4. *Products:* $f \cdot g$
5. *Quotients:* f/g, provided $g(c) \neq 0$
6. *Powers:* f^n, n a positive integer
7. *Roots:* $\sqrt[n]{f}$, provided it is defined on an open interval containing c, where n is a positive integer

2.2.1 Continuous Extension to a Point

Example 24: Show that

$$f(x) = \frac{x^2+x-6}{x^2-4}, \quad x \neq 2$$

has a continuous extension to $x = 2$, and find that extension.

Solution:
Although $f(2)$ is not defined, if $x \neq 2$ we have

$$f(x) = \frac{x^2+x-6}{x^2-4} = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x+2}$$

The new function

$$F(x) = \frac{x+3}{x+2}$$

is equal to $f(x)$ for $x \neq 2$ but is continuous at $x = 2$, having there the value of 5/4. Thus F is the continuous extension of f to $x = 2$, and:

$$\lim_{x\to 2} \frac{x^2+x-6}{x^2-4} = \lim_{x\to 2} f(x) = \frac{5}{4}$$

$$f(x) = \begin{cases} \dfrac{x^2+x-6}{x^2-4}, & x \neq 2 \\ \dfrac{5}{4}, & x = 2 \end{cases}$$

This form is called the continuous extension of the original function to the $x = 2$.