Automatic Control systems

Introduction:

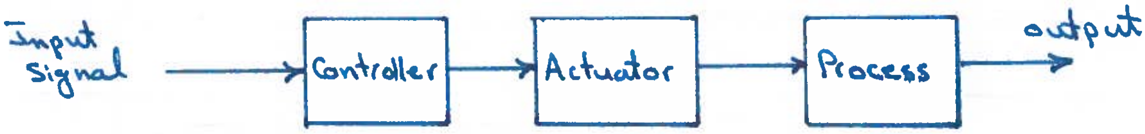
Engineering is concerned with understanding and controlling the materials and forces of nature for the benefit of humankind. Control system engineers are concerned with understanding the controlling segments of their environment, ofen called systems, to provide useful economic products for society. The present challenge to control engineers is the modeling and control of modern, complex, interrelated systems such as traffic control systems, chemical processes, and robotic systems as well many useful and interesting industrial automation systems.

Control engineering integrates the concepts of network theory and communication theory, therefore it is not limited to any engineering discipline but equally applicable to chemical, mechanical, environmental, civil, and electrical engineering.

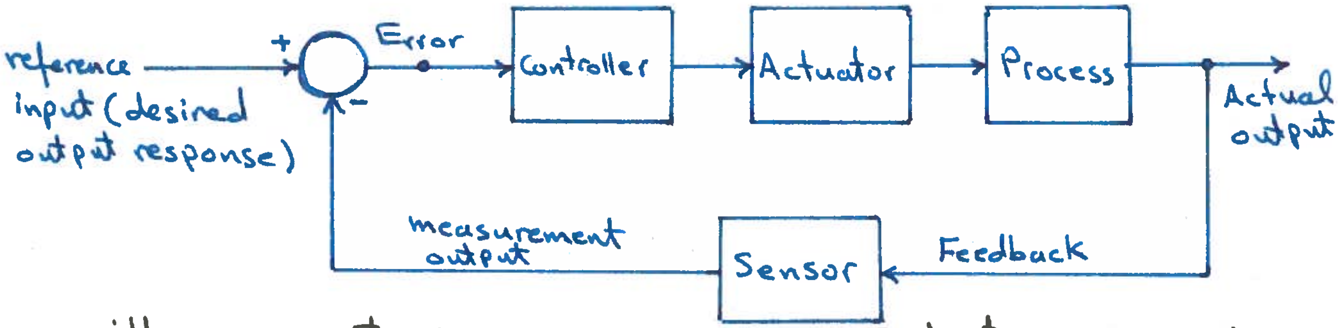
A control system is an interconnection of components forming a system configuration that will provide a desired system response. A component or process to be controlled can be represented by a block.

The input_output relationship represents the cause_and_effect relationship of the process, which in turn represents a processing of the input signal to provide an output signal variable, often with power amplification.

An Open_loop Control system uses a controller and an actuator to obtain the desired response. It is a system without feedback.

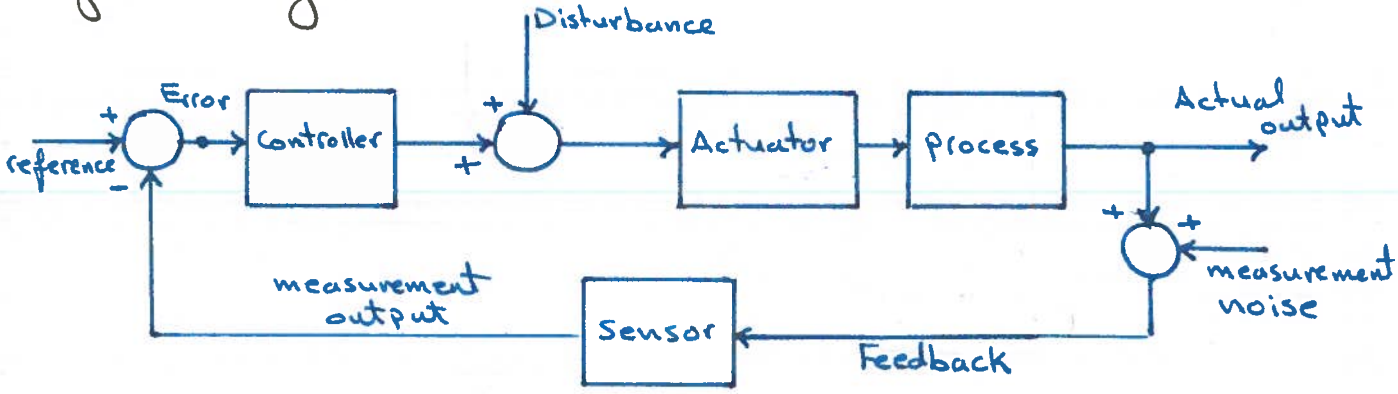


A closed-loop feedback control system utilizes an additional measure of the actual output to compare the actual output with the desired output response (input signal).



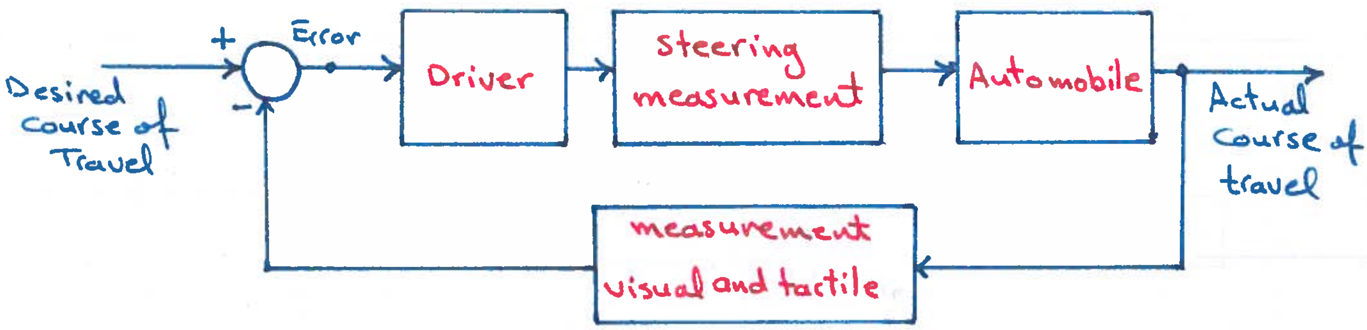
with an acurate sensor, the measured output is a good approximation of the actual output of the system.

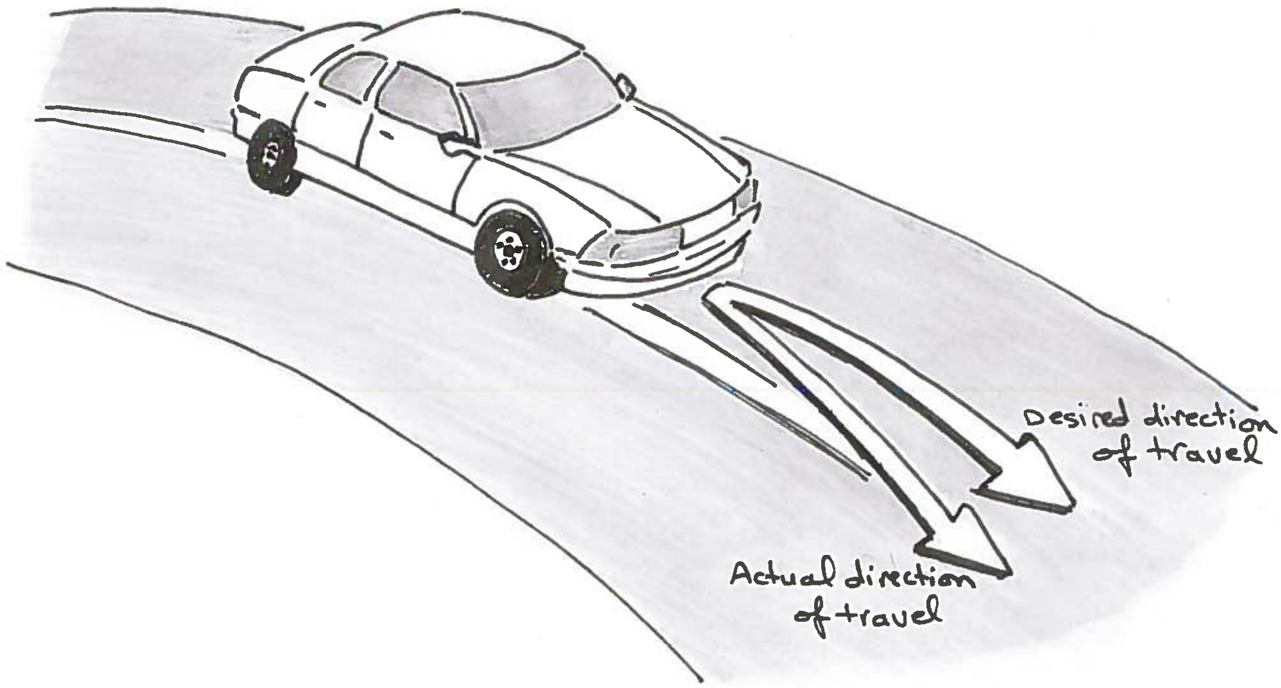
Closed loop control has many advantages over open loop control including the ability to reject external disturbances and improve measurement noise attenuation. External disturbances and measurement noise in real world must be addressed in practical system designs.

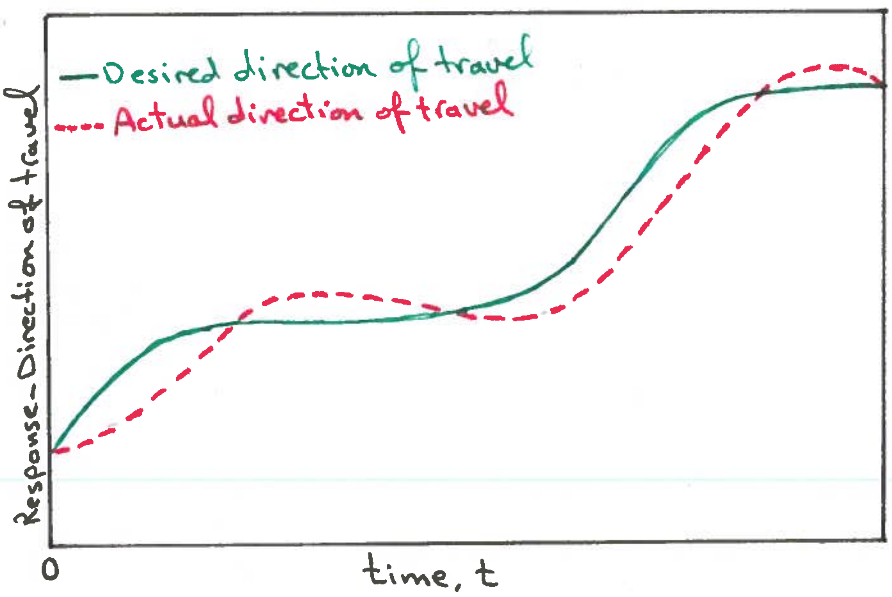


Example of A control system:

A simple block diagram of an automobile steering control system is shown below:







The desired course is compared with measurement of the actual course in order to generate a measure of the error. This measurement is obtained by visual and tactile (body movement) feedback, as provided by the feel of the steering wheel by the hand (Sensor).

Definitions :

Controlled Variable : is the quantity or condition that is measured and controlled.

Control signal / manipulated variable : is the quantity or condition that is varied by the Controller so as to affect the value of the controlled Variable.

Control: measuring the value of the controlled variable of the system and applying the control signal to the system to correct or limit deviation of the measured value from a desired value.

Plants: Any physical object to be controlled, such as mechanical device, heating furnace, chemical reactor, or a spacecraft.

Process: A continuing operation marked by a series of gradual changes and lead to a particular result. it consists of a series of controlled actions.

Systems: A system is a combination of components that act together and perform a certain task.

Disturbance: A disturbance is a signal that tends to affect the value of the output of the system.

Feedback control: It refers to an operation that tends to reduce the difference between the output of a system and same reference input.

Open-loop control system: is one in which the control action is independent of the output.

closed-loop control system: is one in which the control action is somehow dependent on the output.

Bandwidth: The bandwidth of a system is a frequency response measure of how well the system responds to (or filters) variations (or frequencies) in the input signal.

Stable system: The definition of a stable system can be based upon the response of the system to bounded inputs, that is, inputs whose magnitudes are less than some finite value for all time.

A continuous or discrete-time system is said to be stable if every bounded input produces a bounded output.

Major advantages of Open-loop Control system:

- Simple construction and ease of maintenance.
- less expensive than closed loop system.
- no stability problems.
- Convenient when output is hard to measure.

Major disadvantages of open-loop system:

- Disturbances cause errors and output may differ from the desired.
- recalibration is necessary from time to time.

Advantages of Closed-loop Control system:

- Ability to minimize the effect of external disturbance & noise.
- the use of feedback makes the system response insensitive to external disturbance.
- Can use inacurate components to obtain the accurate control of a given plant.

Disadvantages of closed-loop system:

- Stability is a major problem.
- more expensive than open loop system.

The LAPLACE TRANSFORM:

The laplace transform method substitutes relatively easily solved algebric equations for the more difficult differential equations. The time-response solution is obtained by the following operations:

1. obtain the linearized differential equations.
2. obtain the laplace transformation of the differential equations.

3- Solve the resulting algebric equation for the transform of the variable of interest.

The Laplace transformation for a function of time, f(t), is:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

The inverse Laplace transform is written as:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)\, e^{st}\, ds$$

σ : is real positive

example: Find F(s) for $f(t) = e^{-t}$

sol: $\mathcal{L}\{e^{-t}\} = \int_0^{\infty} e^{-t}\, e^{-st}\, dt = \frac{-1}{s+1}\, e^{-(s+1)t}\Big|_0^{\infty} = \frac{1}{s+1}$

Some properties of Laplace Transform

1- if a_1 and a_2 constants,

$$\mathcal{L}(a_1 f_1(t) + a_2 f_2(t)) = a_1 F_1(s) + a_2 F_2(s)$$

2- Transfor of a derivative df/dt,

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0)$$

where $f(0)$ is the initial value of $f(t)$

3- for integral $\int_0^t f(\tau)\, d\tau$

$$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{F(s)}{s}$$

4- The initial value $f(0)$ of a function $f(t)$,

$$f(0) = \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) \quad , t>0$$

This relation is called the initial value Theorem

5- The final value $f(\infty)$ of the function $f(t)$,

$$f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

which is called final value theorem

6- for a Time scaling function, $f(t/a)$,

$$\mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = a\,F(as)$$

7- for a frequency scaling function $F(s/a)$,

$$\mathcal{L}^{-1}\left\{F\left(\frac{s}{a}\right)\right\} = a f(at)$$

8- Time delay function, $f(t-T)$

$$\mathcal{L}\left\{f(t-T)\right\} = e^{-sT} F(s)$$

9- for a function $e^{-at} f(t)$,

$$\mathcal{L}\left\{e^{-at} f(t)\right\} = F(s+a)\text{ , Complex translation}$$

10- if $\mathcal{L}\, f_1(t) = F_1(s)$ and $\mathcal{L}\, f_2(t) = F_2(s)$

then:

$$\mathcal{L}^{-1}\left\{F_1(s) . F_2(s)\right\} = \int_0^t f_1(t) f_2(t-\tau)\, d\tau$$

$$= \int_0^t f_2(\tau) f_1(t-\tau)\, d\tau$$

which is convolution integrals

Table A–1 Laplace Transform Pairs

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{s^n}$
5	$t^n \quad (n = 1, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!} t^{n-1}e^{-at} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at} \quad (n = 1, 2, 3, \ldots)$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
11	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
12	$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
13	$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
14	$\frac{1}{a}(1 - e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(be^{-bt} - ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}\left[1 + \frac{1}{a-b}(be^{-at} - ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$

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Table A–1 *(continued)*

18	$\frac{1}{a^2}(1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s + a)^2}$
19	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s + a)}$
20	$e^{-at} \sin \omega t$	$\frac{\omega}{(s + a)^2 + \omega^2}$
21	$e^{-at} \cos \omega t$	$\frac{s + a}{(s + a)^2 + \omega^2}$
22	$\frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1 - \zeta^2}\, t \quad (0 < \zeta < 1)$	$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
23	$-\frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1 - \zeta^2}\, t - \phi)$ $\phi = \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta}$ $(0 < \zeta < 1, \quad 0 < \phi < \pi/2)$	$\frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
24	$1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1 - \zeta^2}\, t + \phi)$ $\phi = \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta}$ $(0 < \zeta < 1, \quad 0 < \phi < \pi/2)$	$\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$
25	$1 - \cos \omega t$	$\frac{\omega^2}{s(s^2 + \omega^2)}$
26	$\omega t - \sin \omega t$	$\frac{\omega^3}{s^2(s^2 + \omega^2)}$
27	$\sin \omega t - \omega t \cos \omega t$	$\frac{2\omega^3}{(s^2 + \omega^2)^2}$
28	$\frac{1}{2\omega} t \sin \omega t$	$\frac{s}{(s^2 + \omega^2)^2}$
29	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
30	$\frac{1}{\omega_2^2 - \omega_1^2}(\cos \omega_1 t - \cos \omega_2 t) \qquad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2 + \omega_1^2)(s^2 + \omega_2^2)}$
31	$\frac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$	$\frac{s^2}{(s^2 + \omega^2)^2}$

Table A–2 Properties of Laplace Transforms

1	$\mathcal{L}[Af(t)] = AF(s)$
2	$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
3	$\mathcal{L}_\pm\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0\pm)$
4	$\mathcal{L}_\pm\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0\pm) - \dot{f}(0\pm)$
5	$\mathcal{L}_\pm\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}\overset{(k-1)}{f}(0\pm)$ where $\overset{(k-1)}{f}(t) = \frac{d^{k-1}}{dt^{k-1}}f(t)$
6	$\mathcal{L}_\pm\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t)\,dt\right]_{t=0\pm}$
7	$\mathcal{L}_\pm\left[\int\cdots\int f(t)(dt)^n\right] = \frac{F(s)}{s^n} + \sum_{k=1}^{n}\frac{1}{s^{n-k+1}}\left[\int\cdots\int f(t)(dt)^k\right]_{t=0\pm}$
8	$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$
9	$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s)$ if $\int_0^\infty f(t)\,dt$ exists
10	$\mathcal{L}[e^{-at}f(t)] = F(s+a)$
11	$\mathcal{L}[f(t-\alpha)1(t-\alpha)] = e^{-\alpha s}F(s) \qquad \alpha \geq 0$
12	$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$
13	$\mathcal{L}[t^2f(t)] = \frac{d^2}{ds^2}F(s)$
14	$\mathcal{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \qquad (n = 1, 2, 3, \dots)$
15	$\mathcal{L}\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s)\,ds$ if $\lim_{t\to 0}\frac{1}{t}f(t)$ exists
16	$\mathcal{L}\left[f\left(\frac{1}{a}\right)\right] = aF(as)$
17	$\mathcal{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau\right] = F_1(s)F_2(s)$
18	$\mathcal{L}[f(t)g(t)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)G(s-p)\,dp$

Initial value theorem	$f(0+) = \lim_{t \to 0+} f(t) = \lim_{s \to \infty} sF(s)$
Final value theorem	$f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$
Pulse function $f(t) = \frac{A}{t_0} 1(t) - \frac{A}{t_0} 1(t - t_0)$	$\mathcal{L}[f(t)] = \frac{A}{t_0 s} - \frac{A}{t_0 s} e^{-st_0}$
Impulse function $g(t) = \lim_{t_0 \to 0} \frac{A}{t_0}, \quad$ for $0 < t < t_0$ $= 0, \quad$ for $t < 0, t_0 < t$	$\mathcal{L}[g(t)] = \lim_{t_0 \to 0} \left[\frac{A}{t_0 s} (1 - e^{-st_0}) \right]$ $= \lim_{t_0 \to 0} \frac{\frac{d}{dt_0}[A(1 - e^{-st_0})]}{\frac{d}{dt_0}(t_0 s)}$ $= \frac{As}{s} = A$

example: Find $\mathcal{L}(3e^{-t} - e^{-2t})$

sol $\mathcal{L}(3e^{-t} - e^{-2t}) = 3\mathcal{L}(e^{-t}) - \mathcal{L}(e^{-2t})$

$$= \frac{3}{s+1} - \frac{1}{s+2} = \frac{2s+5}{s^2+3s+2}$$

example: find $\mathcal{L}^{-1}\left(\frac{2}{s+1} - \frac{4}{s+3}\right)$

sol

$$\mathcal{L}^{-1}\left(\frac{2}{s+1} - \frac{4}{s+3}\right) = 2\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - 4\mathcal{L}^{-1}\left(\frac{1}{s+3}\right)$$
$$= 2e^{-t} - 4e^{-3t}$$

example: Find $\mathcal{L}\left[\frac{d}{dt}(e^{-t})\right]$ if $\lim_{t\to 0} e^{-t} = 1$

sol

$$\mathcal{L}\left[\frac{d}{dt}(e^{-t})\right] = s\left(\frac{1}{s+1}\right) - 1 = \frac{-1}{s+1}$$

example: Find $\mathcal{L}\left[\int_0^t e^{-\tau} d\tau\right]$

sol

$$\mathcal{L}\left[\int_0^t e^{-\tau} d\tau\right] = \frac{1}{s}\left(\frac{1}{s+1}\right) = \frac{1}{s(s+1)}$$

example: find the initial value of $f(t) = e^{-3t}$

sol $F(s) = \mathcal{L} f(t) = \frac{1}{s+3}$

$$\rightarrow \lim_{t\to 0} e^{-3t} = \lim_{s\to\infty} s \frac{1}{s+3} = \lim_{s\to\infty} \cancel{s} \frac{1}{\cancel{s}(1+\frac{3}{s})} = 1$$

example: if $f(t) = 1 - e^{-t}$, find the final value of this function,

sol $F(s) = \mathcal{L}(1 - e^{-t}) = \frac{1}{s} - \frac{1}{s+1} = \frac{1}{s(s+1)}$

$$\rightarrow \lim_{t\to\infty} (1 - e^{-t}) = \lim_{s\to 0} \cancel{s} \frac{1}{\cancel{s}(s+1)} = 1$$

example: Find $\mathcal{L}(\cos t)$

sol $F(s) = \mathcal{L}[\cos t] = \frac{s}{s^2+1}$

example: Find $\mathcal{L}[e^{-2t}\cos t]$

sol $\mathcal{L}[e^{-2t}\cos t] = \dfrac{s+2}{(s+2)^2+1} = \dfrac{s+2}{s^2+4s+5}$

example: $F(s) = \dfrac{s}{(s+1)(s^2+1)}$; find $f(t)$,

sol

$$F_1(s) = \frac{1}{s+1} \longrightarrow f_1(t) = e^{-t}$$

$$\longrightarrow f_1(t-\tau) = e^{-(t-\tau)}$$

$$F_2(s) = \frac{s}{s^2+1} \longrightarrow f_2(t) = \cos t$$

$$\longrightarrow f_2(\tau) = \cos\tau$$

$$\rightarrow \mathcal{L}^{-1}\left[\frac{1}{s+1}\cdot\frac{s}{s^2+1}\right] = \int_0^t e^{-(t-\tau)} \cos\tau \, d\tau$$

$$= e^{-t}\int_0^t e^{\tau}\cos\tau \, d\tau$$

$$= \frac{1}{2}\left(\cos t + \sin t - e^{-t}\right)$$

Transfer Function :

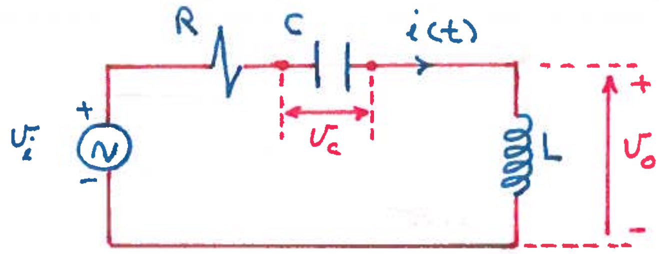
The transfer function of a linear time-invariant differential equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under assumption that all initial Conditions are Zero.

$G(s)$ = Transfer function

$$= \left.\frac{\mathcal{L}\, y(t)}{\mathcal{L}\, r(t)}\right|_{\text{zero initial conditions}}$$

$$\rightarrow G(s) = \frac{Y(s)}{R(s)}$$

example : Consider the below electrical network, find the transfer function $\frac{V_o}{V_i}$,



sol

$$v_i = iR + v_c + L\frac{di}{dt}$$

taking Laplace trasform

$$\rightarrow V_i = IR + V_c + LSI \quad \dots (1)$$, assuming zero initial Conditions

now, $i = c\frac{dv_c}{dt}$

$$\rightarrow I = CSV_c \rightarrow V_c = \frac{I}{CS} \quad \dots (2)$$

sub. in (1)

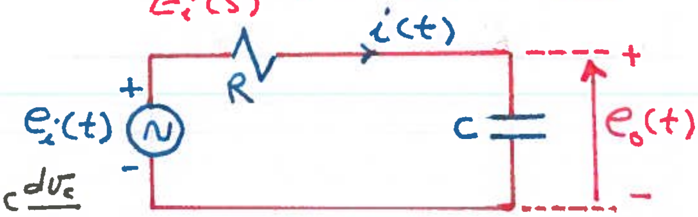
$$\rightarrow V_i = IR + \frac{I}{CS} + LSI$$

$$v_o = L\frac{di}{dt} \rightarrow V_o = LSI$$

$$\rightarrow \text{Transfer function} = \frac{V_o}{V_i} = \frac{LSI(s)}{I(s)[R+LS+\frac{1}{CS}]}$$

$$\rightarrow \frac{V_o}{V_i} = \frac{S^2}{S^2 + \frac{R}{L}S + \frac{1}{LC}}$$

example : Find the transfer function $\frac{E_o(S)}{E_i(S)}$ for the circuit shown ,



sol

$$e_i(t) = i(t)R + v_c(t) \;;\; \text{and } i(t) = c\frac{dv_c}{dt}$$

$$\rightarrow E_i(s) = I(s)R + \frac{I(s)}{CS} \quad \dots (1)$$

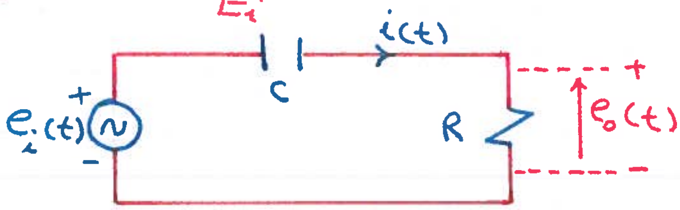
$$\text{and } E_o = V_c(s) = \frac{I(s)}{CS} \quad \dots (2)$$

$$\rightarrow \text{Transfer function} = \frac{E_o(S)}{E_i(S)} = \frac{\frac{I(s)}{CS}}{I(s)R + \frac{I(s)}{CS}}$$

$$= \frac{1}{RCS+1} = \frac{1}{TS+1}$$

where $T = RC$

example: Find the transfer function $\frac{E_o}{E_i}$ for the circuit shown,



<u>sol</u>

$e_i(t) = V_c(t) + i(t)R$

$i(t) = C\frac{dV_c}{dt} \rightarrow I(s) = CsV_c(s)$, Zero initial conditions

$\rightarrow V_c(s) = I(s)/Cs$

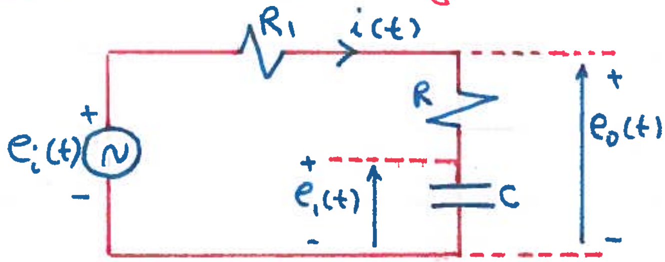
$\rightarrow E_i(s) = \frac{I(s)}{Cs} + I(s)R = I(s)\left[R + \frac{1}{Cs}\right]$ ---- (1)

$e_o(t) = i(t)R \rightarrow E_o(s) = I(s)\cdot R$ ---- (2)

$\rightarrow$ Transfer function $= \frac{E_o(s)}{E_i(s)} = \frac{I(s)\cdot R}{I(s)\left[R+\frac{1}{Cs}\right]} = \frac{RCs}{RCs+1} = \frac{Ts}{Ts+1}$

where $T = RC$

example: For the circuit shown below, find the voltage transfer function,



<u>sol</u>

$e_i(t) = i(t)R_1 + i(t)R + e_1(t)$

$= i(t)(R_1+R) + \frac{1}{C}\int i(t)dt$

$E_i(s) = I(s)[R_1+R] + \frac{1}{Cs}I(s)$, Zero initial Conditions

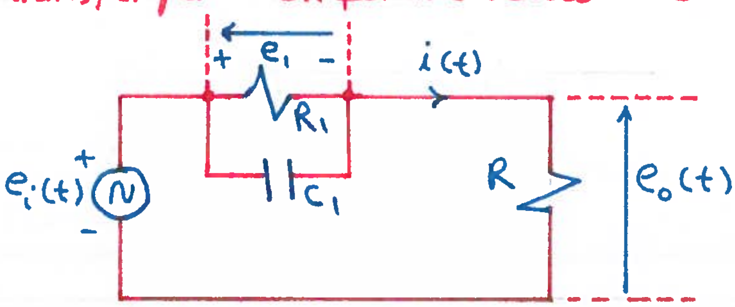
$E_i(s) = I(s)\left[R+R_1+\frac{1}{Cs}\right]$ ---- (1)

$e_o(t) = i(t)R + e_1(t) = i(t)R + \frac{1}{C}\int i(t)dt$

$\rightarrow E_o(s) = I(s)R + \frac{1}{Cs}I(s) = I(s)\left[R + \frac{1}{Cs}\right]$ ---- (2)

$\rightarrow \frac{E_o}{E_i} = \frac{I(s)\left[R+\frac{1}{Cs}\right]}{I(s)\left[R+R_1+\frac{1}{Cs}\right]} = \frac{RCs+1}{RCs[R+R_1]+1}$

example: Find the transfer function for the below circuit,



sol $e_o(t) = i(t)R \rightarrow E_o(s) = I(s)R$ --- ①

$e_i(t) = e_1(t) + i(t)R$

$$e_1(t) = i(t) * Z_1 \rightarrow E_1(s) = I(s) * \frac{R_1 \cdot \frac{1}{C_1 S}}{R + \frac{1}{C_1 S}}$$

$$= I(s) \cdot \frac{R_1}{RC_1 S + 1}$$

$$\rightarrow E_i(s) = I(s) \cdot \frac{R_1}{RC_1 S + 1} + I(s)R \quad \text{---} \quad ②$$

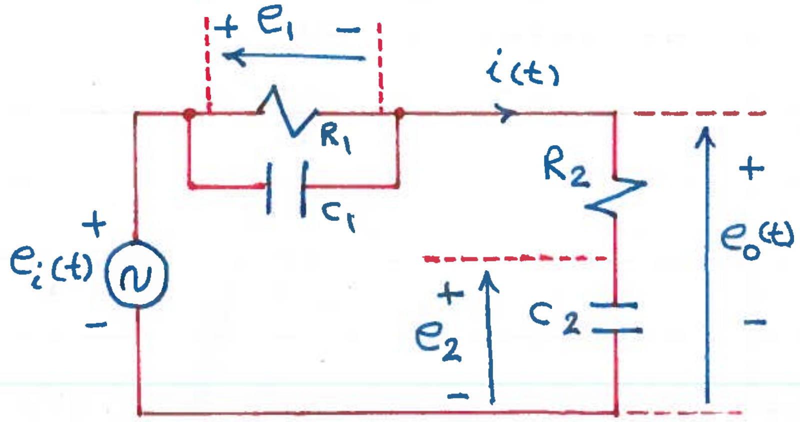
$$\rightarrow \text{Transfer function} = \frac{E_o}{E_i} = \frac{I(s)\ R}{I(s) \cdot \left[\frac{R_1}{RC_1 S + 1} + R\right]}$$

$$= \frac{R(RC_1 S + 1)}{R(RC_1 S + 1) + R_1}$$

example: Find the Transfor function for the below circuit,

sol

note that $e_o = e_i \dfrac{Z_2}{Z_1 + Z_2}$



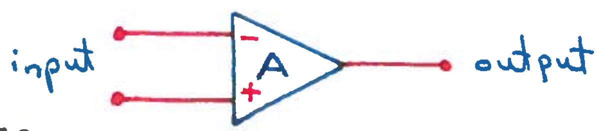
$$\rightarrow E_o = E_i \frac{Z_2(S)}{Z_1(S) + Z_2(S)}$$

$$\frac{E_o}{E_i} = \frac{R_2 + \frac{1}{C_2 S}}{R_2 + \frac{1}{C_2 S} + \frac{R_1 / C_1 S}{R_1 + \frac{1}{C_1 S}}}$$

$$\rightarrow \frac{E_o}{E_i} = \frac{(R_1 R_2 C_1 C_2) S^2 + (R_1 C_1 + R_2 C_2) S + 1}{(R_1 R_2 C_1 C_2) S^2 + (R_1 C_1 + R_2 C_2 + R_1 R_2 C_2) S + 1}$$

Operational Amplifiers (op amp):

Ideal operational Amplifier has the following characteristics:



1- ifinite input impedance

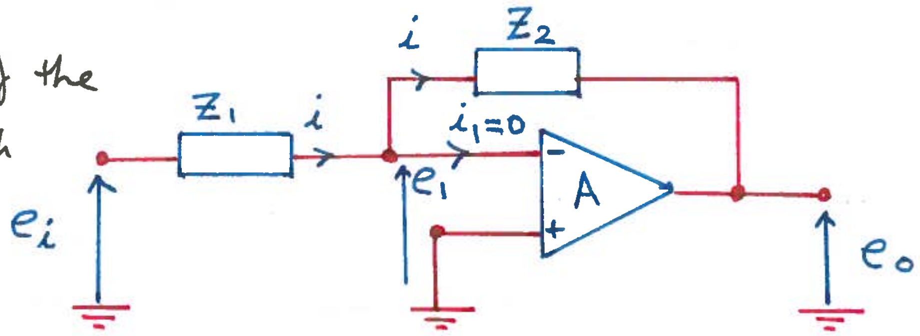
2- infinite gain

3- Zero output impedance

4- infinite Bandwidth

Let's take the following circuit for example:

input impedance of the op amp is very high therefore $i_1 = 0$



$$I(s) = \frac{E_i(s) - E_1(s)}{Z_1(s)} \qquad \frac{E_1(s) - E_o(s)}{Z_2(s)}$$

E_1 is virtually Zero, because $i_1 = 0$ & infinite gain of op amp

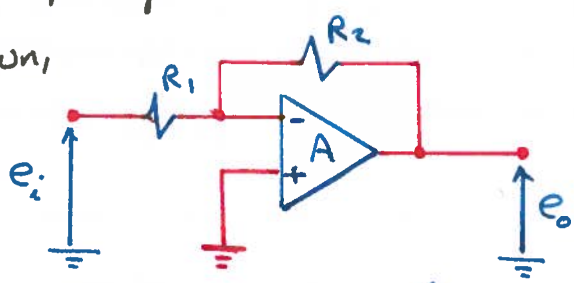
$$\rightarrow I(s) = \frac{E_i(s)}{Z_1(s)} = -\frac{E_o(s)}{Z_2(s)}$$

or
$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_2}{Z_1}$$

where Z_2 is the impedance connected between the output and input of the op amp., and Z_1 is the impedance connected to the input of the op amp.

For $Z_2 = R_2$, and $Z_1 = R_1$, as shown,

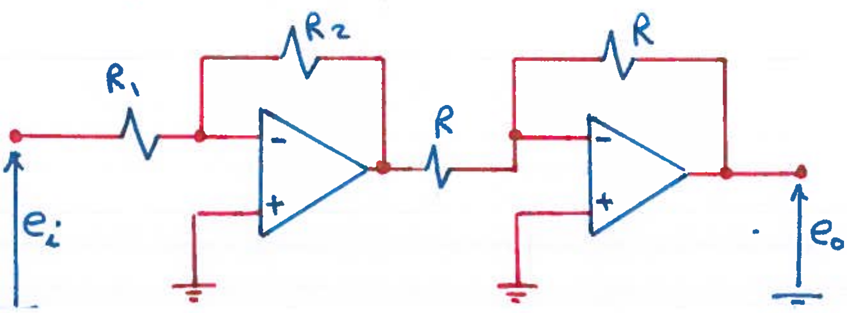
thus
$$\frac{E_o(s)}{E_i(s)} = -\frac{R_2}{R_1}$$



-Inverting Amplifier-

and the circuit is called an Inverting Amplifier

example: Find the Transfer function for the below circuit,



sol

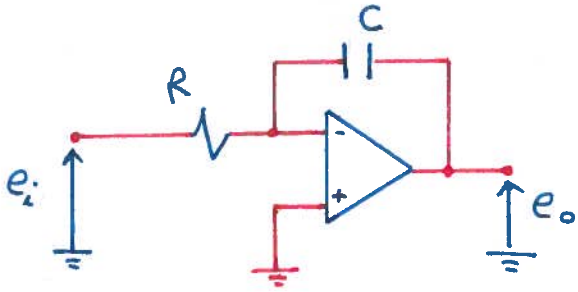
$$\frac{E_o}{E_i} = -\frac{R_2}{R_1} \cdot \left(-\frac{R}{R}\right)$$

$$\rightarrow \frac{E_o}{E_i} = \frac{R_2}{R_1}$$

example: Find $e_o(t)$ for the circuit shown below,

Sol

$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_2}{Z_1} = -\frac{1/CS}{R}$$
$$= -\frac{1}{RCS}$$

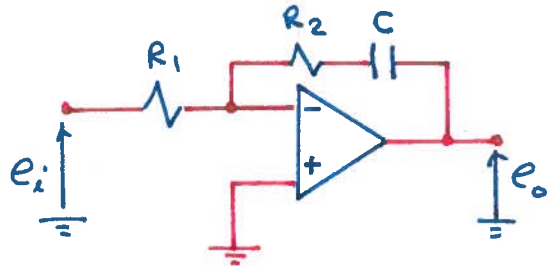


$$E_o(s) = -\frac{1}{RCS} E_i(s)$$

$$\rightarrow e_o(t) = \mathcal{L}^{-1} E_o(s) = -\frac{1}{RC}\int_0^t e_i(\tau)\,d\tau$$

The circuit is an integrator

example: Find $e_o(t)$ for the below circuit and mention how it Processes the input Signal.



Sol

$$\frac{E_o(s)}{E_i(s)} = -\frac{R_2 + \frac{1}{CS}}{R_1}$$
$$= -\frac{R_2}{R_1} - \frac{1}{R_1 CS}$$

$$E_o(s) = -\frac{R_2}{R_1} E_i(s) - \frac{1}{R_1 CS} E_i(s)$$

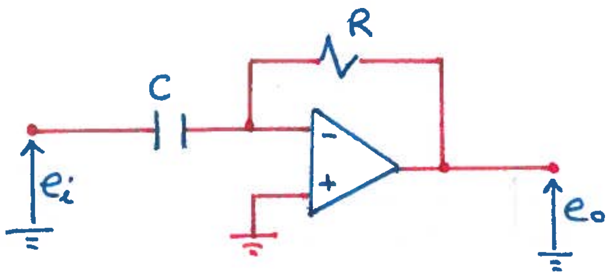
$$\rightarrow e_o(t) = \mathcal{L}^{-1} E_o(s) = -\frac{R_2}{R_1} e_i(t) - \frac{1}{R_1 C}\int_0^t e_i(\tau)\,d\tau$$

The circuit processes the input Signal by Proportional plus integral.

example: Derive the output signal with respect to the input signal for the circuit shown,

Sol

$$\frac{E_o(s)}{E_i(s)} = -\frac{R}{1/CS} = -RCS$$

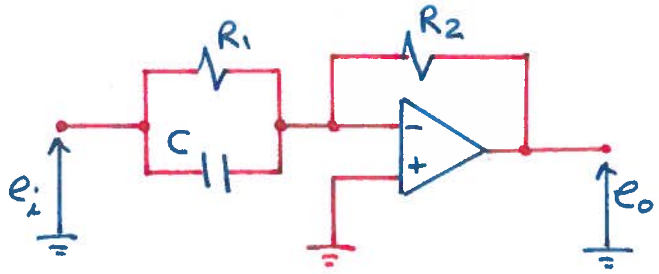


$$E_o(s) = -RCS\,E_i(s)$$

$$\rightarrow e_o(t) = -RC\frac{de_i(t)}{dt}$$

The Circuit is a differentiator

example: Find the output Voltage with respect to time and Comment on how it processes the input signal, for the below circuit.



Sol

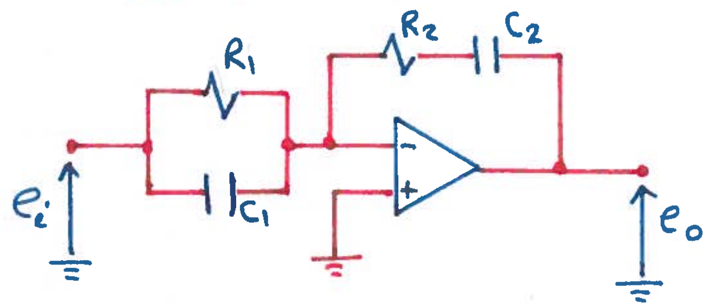
$$\frac{E_o(s)}{E_i(s)} = -\frac{R_2}{\dfrac{R_1 \cdot \frac{1}{Cs}}{R_1 + \frac{1}{Cs}}} = -\frac{R_2}{R_1}(R_1 C s + 1)$$

$$E_o(s) = -\frac{R_2}{R_1} E_i(s) - R_2 C s\, E_i(s)$$

$$\rightarrow e_o(t) = -\frac{R_2}{R_1} e_i(t) - R_2 C \frac{d e_i(t)}{dt}$$

The circuit processes the input signal by proportional and derivative actions.

example: Find $e_o(t)$ for the below circuit then comment.



Sol

$$\frac{E_o}{E_i} = -\frac{R_2 + \frac{1}{C_2 s}}{\dfrac{R_1 \cdot \frac{1}{C_1 s}}{R_1 + \frac{1}{C_1 s}}}$$

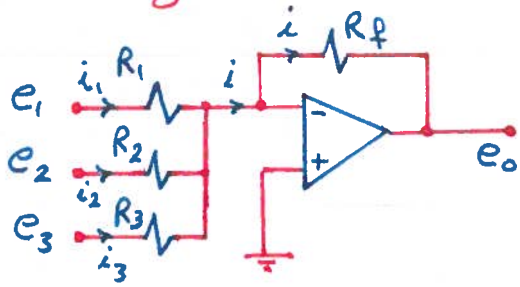
$$= -\frac{R_1 R_2 C_1 C_2 s^2 + R_1 C_1 s + R_2 C_2 s + 1}{R_1 C_2 s}$$

$$\rightarrow E_o = -\frac{R_1 C_1 + R_2 C_2}{R_1 C_2} E_i - \frac{1}{R_1 C_2 s} E_i - R_2 C_1 s\, E_i$$

$$\rightarrow e_o(t) = -\frac{R_1 C_1 + R_2 C_2}{R_1 C_2} e_i(t) - \frac{1}{R_1 C_2}\int_0^t e_i(\tau)\,d\tau - R_2 C_1 \frac{d e_i(t)}{dt}$$

$\therefore$ The circuit processes the input signal with proportional and integral and derivative actions.

example: what is the function of the following circuit?



sol

$$i = i_1 + i_2 + i_3$$
$$= \frac{e_1}{R_1} + \frac{e_2}{R_2} + \frac{e_3}{R_3}$$

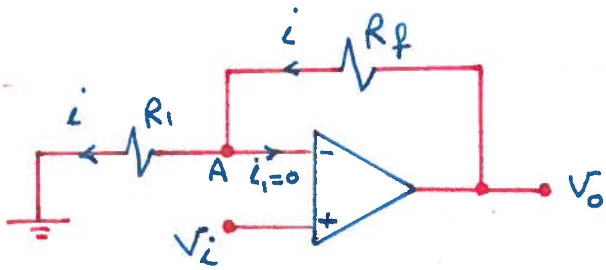
$$e_o = -i R_f = -\left(\frac{e_1}{R_1} + \frac{e_2}{R_2} + \frac{e_3}{R_3}\right) R_f$$

or $$E_o = -\frac{R_f}{R_1} E_1 - \frac{R_f}{R_2} E_2 - \frac{R_f}{R_3} E_3$$

it is a summing amplifier

non Inverting Amplifier

since $i_1 = 0$



$\rightarrow V_A \approx V_i$

$$\rightarrow i = \frac{V_o - V_A}{R_f} = \frac{V_A}{R_1}$$

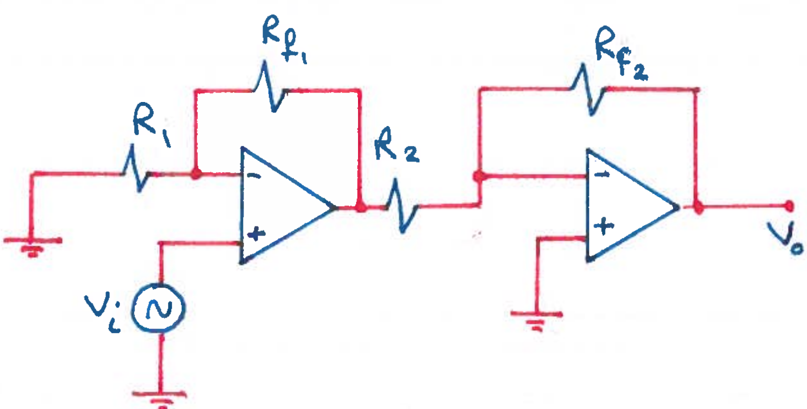
$$\rightarrow \frac{V_o - V_i}{R_f} = \frac{V_i}{R_1} \rightarrow V_i (R_f + R_1) = V_o R_1$$

$$\rightarrow \text{gain} = A = \frac{V_o}{V_i} = \frac{R_1 + R_f}{R_1} = 1 + \frac{R_f}{R_1}$$, (non-inverting Amplifier)

example: Find the output voltage in the below circuit,

sol

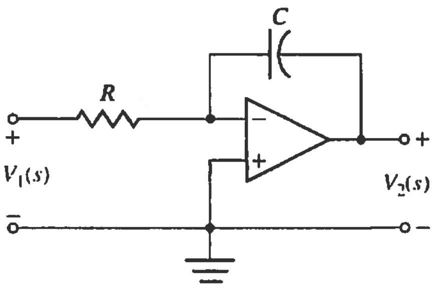
$$V_o = V_i \left(1 + \frac{R_{f_1}}{R_1}\right) \cdot \left(-\frac{R_{f_2}}{R_2}\right)$$



Transfer Functions of Dynamic Elements and Networks

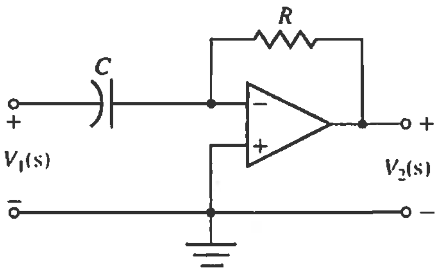
Element or System	$G(s)$

1. Integrating circuit, filter



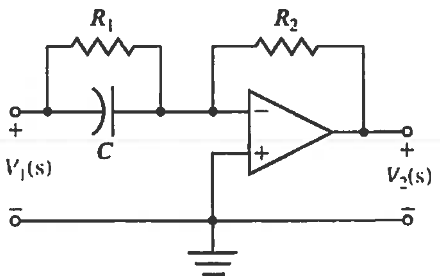
$$\frac{V_2(s)}{V_1(s)} = -\frac{1}{RCs}$$

2. Differentiating circuit



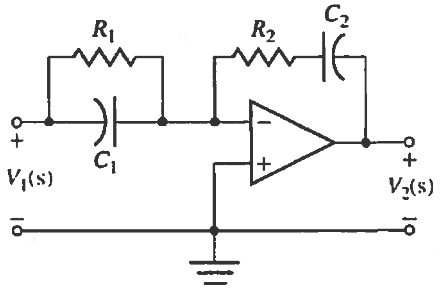
$$\frac{V_2(s)}{V_1(s)} = -RCs$$

3. Differentiating circuit



$$\frac{V_2(s)}{V_1(s)} = -\frac{R_2(R_1Cs + 1)}{R_1}$$

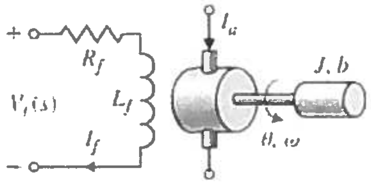
4. Integrating filter



$$\frac{V_2(s)}{V_1(s)} = -\frac{(R_1C_1s + 1)(R_2C_2s + 1)}{R_1C_2s}$$

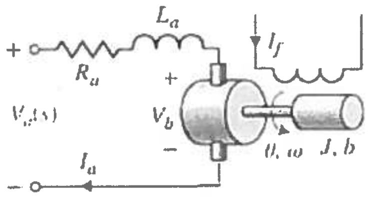
Element or System	$G(s)$

5. DC motor, field-controlled, rotational actuator



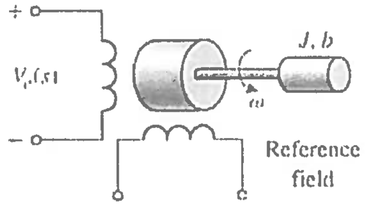
$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(Js + b)(L_f s + R_f)}$$

6. DC motor, armature-controlled, rotational actuator



$$\frac{\theta(s)}{V_a(s)} = \frac{K_m}{s[(R_a + L_a s)(Js + b) + K_b K_m]}$$

7. AC motor, two-phase control field, rotational actuator

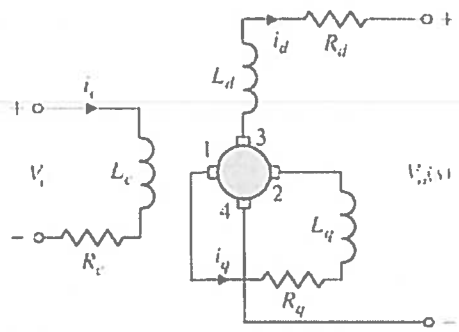


$$\frac{\theta(s)}{V_c(s)} = \frac{K_m}{s(\tau s + 1)}$$

$$\tau = J/(b - m)$$

m = slope of linearized torque-speed curve (normally negative)

8. Rotary Amplifier (Amplidyne)



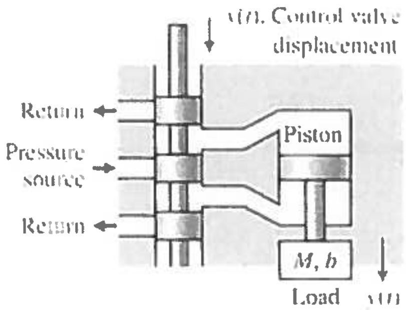
$$\frac{V_o(s)}{V_c(s)} = \frac{K/(R_c R_q)}{(s\tau_c + 1)(s\tau_q + 1)}$$

$$\tau_c = L_c/R_c, \qquad \tau_q = L_q/R_q$$

for the unloaded case, $i_d \approx 0$, $\tau_c \approx \tau_q$, $0.05\ \text{s} < \tau_c < 0.5\ \text{s}$

$$V_q,\ V_{34} = V_d$$

9. Hydraulic actuator



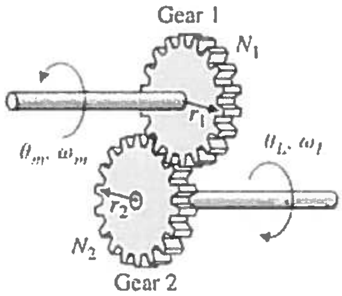
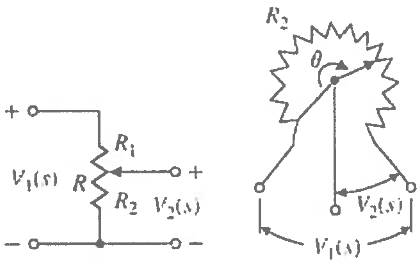
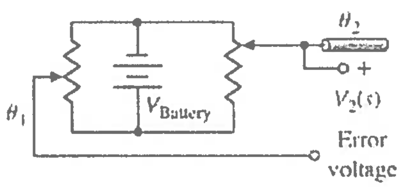
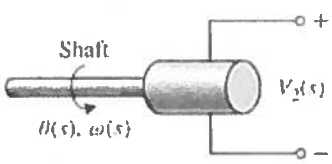
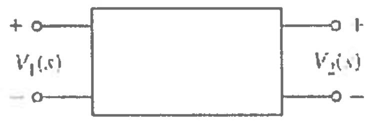
$$\frac{Y(s)}{X(s)} = \frac{K}{s(Ms + B)}$$

$$K = \frac{Ak_x}{k_p}, \qquad B = \left(b + \frac{A^2}{k_p}\right)$$

$$k_x = \left.\frac{\partial g}{\partial x}\right|_{x_0}, \qquad k_p = \left.\frac{\partial g}{\partial P}\right|_{P_0},$$

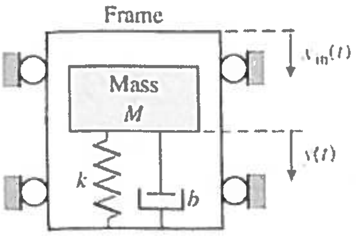
$$g = g(x, P) = \text{flow}$$

$$A = \text{area of piston}$$

Element or System	G(s)
10. Gear train, rotational transformer 	Gear ratio $= n = \dfrac{N_1}{N_2}$ $N_2\theta_L = N_1\theta_m, \quad \theta_L = n\theta_m$ $\omega_L = n\omega_m$
11. Potentiometer, voltage control 	$\dfrac{V_2(s)}{V_1(s)} = \dfrac{R_2}{R} = \dfrac{R_2}{R_1 + R_2}$ $\dfrac{R_2}{R} = \dfrac{\theta}{\theta_{\max}}$
12. Potentiometer, error detector bridge 	$V_2(s) = k_s(\theta_1(s) - \theta_2(s))$ $V_2(s) = k_s\theta_{\text{error}}(s)$ $k_s = \dfrac{V_{\text{Battery}}}{\theta_{\max}}$
13. Tachometer, velocity sensor 	$V_2(s) = K_t\omega(s) = K_t s\theta(s)$ K_t = constant
14. DC amplifier 	$\dfrac{V_2(s)}{V_1(s)} = \dfrac{k_a}{s\tau + 1}$ R_o = output resistance C_o = output capacitance $\tau = R_oC_o, \tau \ll 1s$ and is often negligible for controller amplifier

Element or System	G(s)

15. Accelerometer, acceleration sensor



$x_o(t) = y(t) - x_{in}(t)$,

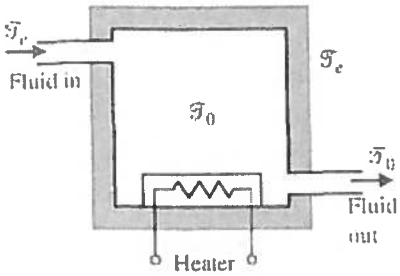
$$\frac{X_o(s)}{X_{in}(s)} = \frac{-s^2}{s^2 + (b/M)s + k/M}$$

For low-frequency oscillations, where

$\omega < \omega_n$,

$$\frac{X_o(j\omega)}{X_{in}(j\omega)} \simeq \frac{\omega^2}{k/M}$$

16. Thermal heating system



$$\frac{\mathcal{T}(s)}{q(s)} = \frac{1}{C_t s + (QS + 1/R_t)}, \text{ where}$$

$\mathcal{T} = \mathcal{T}_o - \mathcal{T}_e$ = temperature difference due to thermal process

C_t = thermal capacitance

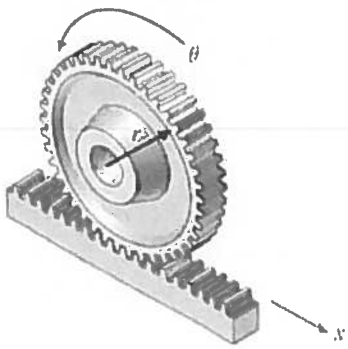
Q = fluid flow rate = constant

S = specific heat of water

R_t = thermal resistance of insulation

$q(s)$ = transform of rate of heat flow of heating element

17. Rack and pinion



$x = r\theta$

converts radial motion

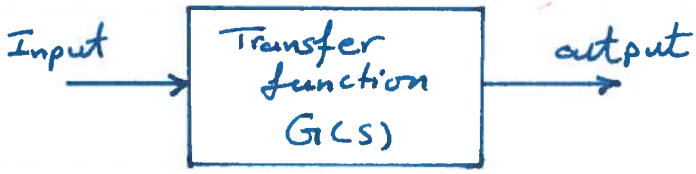
to linear motion

Block Diagrams

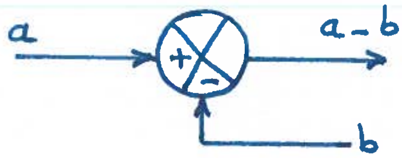
A Block diagram of a system is a pictorial representation of the function performed by each component and the flow of signals.

A block diagram has the advantage of indicating more realistically the signal flows of the actual system.

In a block diagram all system variables are linked to each other through functional blocks. The block is a symbol for the mathematical operation on the input signal to the block that produces the output. The transfer functions of the components are usually entered in the corresponding blocks, which are connected by arrows to indicate the direction of the flow of signals.

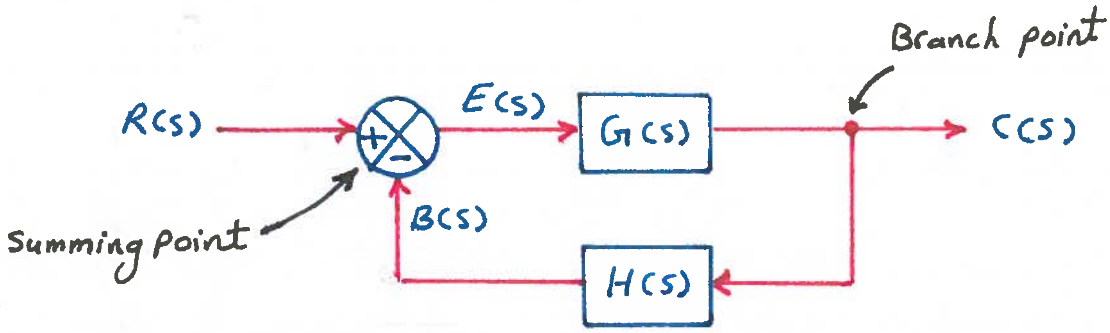


Summing Point: As in figure below, it indicates a summing operation. The plus or minus sign at each arrowhead indicates wheather that the signal is to be added or subtracted. The quantities being added or subtracted must have same dimentions and same units.



Branch point: A branch point is a point from which the signal from a block goes concurrently to other blocks or summing points.

Block Diagram of a closed-loop System :



The output $C(s)$ is fed back to the summing point and compared with the reference input $R(s)$. The output of the block $C(s)$ is obtained by multiplying the transfer function $G(s)$ by the input to the block, $E(s)$. $H(s)$ is the transfer function of the feedback element.

The feedback signal that is feedback to the summing point for comparison with the input is $B(s) = H(s)\,C(s)$

open-loop Transfer function:

$$\text{open-loop transfer function} = \frac{B(s)}{E(s)} = G(s)\,H(s)$$

Feed forward transfer function:

$$\text{Feed Forward Transfer function} = G(s) = \frac{C(s)}{E(s)}$$

closed-loop Transfer function:

$$C(s) = G(s)E(s)$$

$$E(s) = R(s) - H(s)\,C(s)$$

$$\rightarrow C(s) = G(s)\left[R(s) - H(s)\,C(s)\right]$$

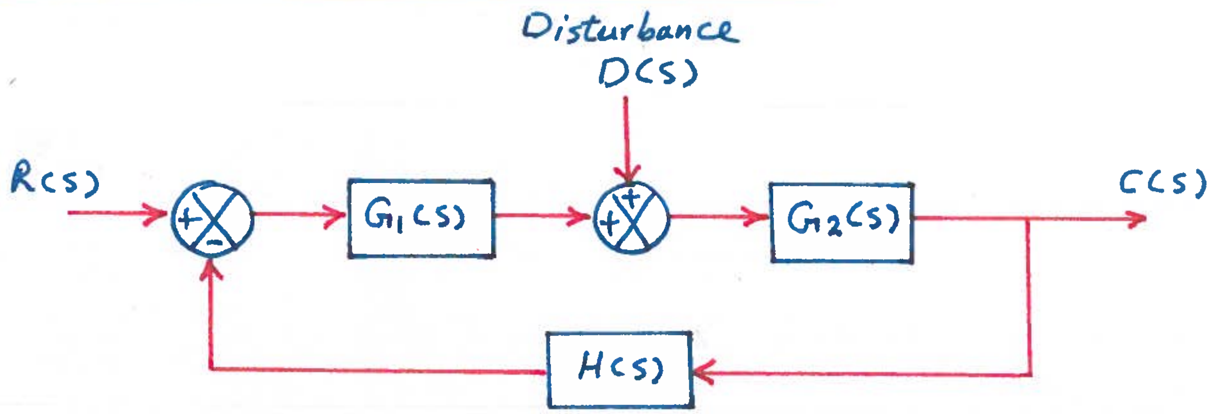
$$C(s) + C(s)\,G(s)\,H(s) = G(s)\,R(s)$$

$$C(s)\left[1 + G(s)\,H(s)\right] = G(s)\,R(s)$$

$$\rightarrow \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

which is called the closed-loop transfer function

closed-loop system subjected to a disturbance:



when two inputs, $R(s)$ and $D(s)$, are present in a linear time-invariant system, each input can be treated independently of the other, and the outputs corresponding to each input alone can be added.

if $C_D(s)$ is the output corresponding to disturbance $D(s)$,

$C_R(s)$ is the output corresponding to the reference input $R(s)$

Thus,

$$\frac{C_D(s)}{D(s)} = \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)}$$

$$\frac{C_R(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H(s)}$$

$$C(s) = C_R(s) + C_D(s) = \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)}\left[G_1(s)R(s) + D(s)\right]$$

when $|G_1(s)H(s)| \gg 1$ and $|G_1(s)G_2(s)H(s)| \gg 1$ then $\frac{C_D(s)}{D(s)}$ becomes almost zero and then the effect of disturbance is suppressed and this is an advantage of the closed-loop system.

If $|G_1(s)G_2(s)H(s)|$ increases and becomes $\gg 1$ then $\frac{C_R(s)}{R(s)}$ approaches $\frac{1}{H(s)}$, here the closed-loop transfer function $\frac{C_R(s)}{R(s)}$ becomes independent of $G_1(s)$ and $G_2(s)$ which is another advantage of the closed-loop system.

It is also seen that any closed-loop system with unity feedback, $H(s) = 1$, tends to equalize the input and output.

Procedure For Drawing a Block Diagram

To draw a block diagram for a system,

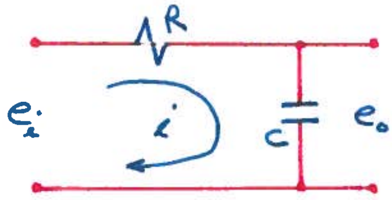
1- Write the equation that describe the dynamic behavior of each component.

2- Take the Laplace transforms of these equations, assuming zero initial conditions, and represent each Laplace-Transformed equation individually in block form.

3- Assemble the elements into a complete block diagram.

example: Consider the RC circuit shown below,

Applying the previous steps,

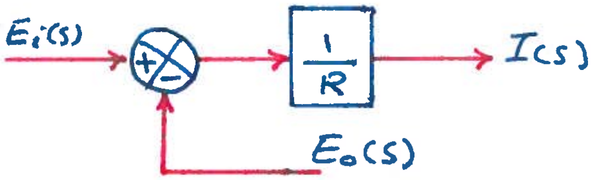


1. the equations for this circuits are:

$$i = \frac{e_i - e_o}{R} \quad , \quad e_o = \frac{1}{c}\int i\,dt$$

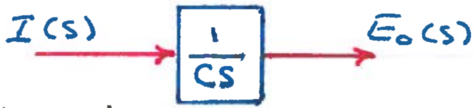
2. taking Laplace transform, and represent each equation in a block form,

$$I(s) = \frac{E_i(s) - E_o(s)}{R}$$

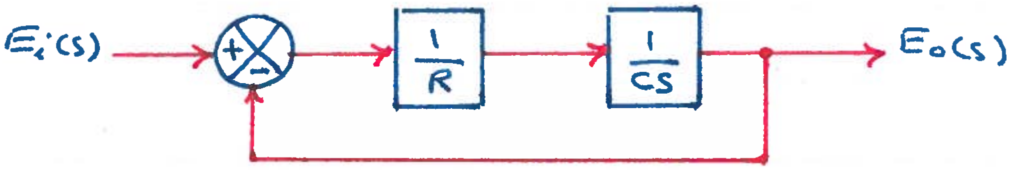


and

$$E_o(s) = \frac{1}{Cs} I(s)$$



3. assembling the above blocks to obtain the final block diagram,



Block Diagram Reduction:

Blocks can be connected in series only if the output of one block is not affected by the next following block. If there are any loading effects between the components, it is necessary to combine these components into a single block.

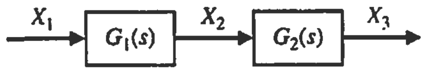
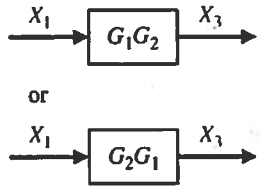
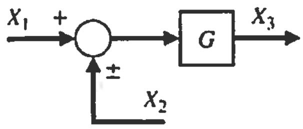
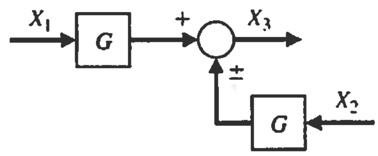
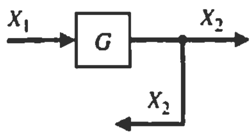
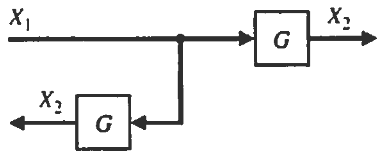
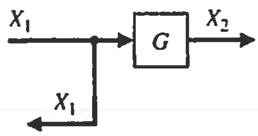
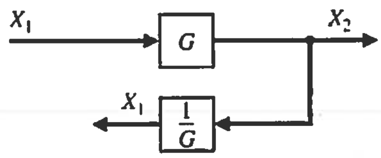
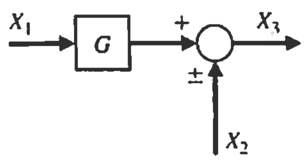
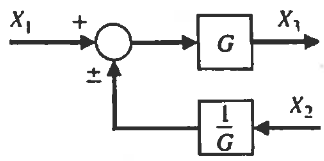
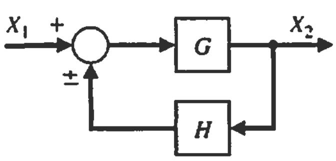
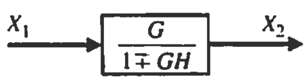
Any number of cascaded blocks representing nonloading components can be replaced by a single block, the transfer function of which is simply the product of the individual transfer functions.

A complicated block diagram involving many feedback loops can be simplified by a step-by-step arrangement. This arrangement reduces the labor needed for subsequent mathematical analysis.

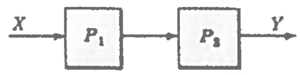
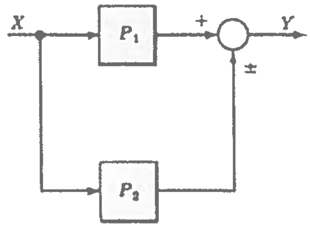
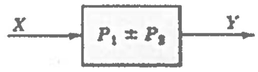
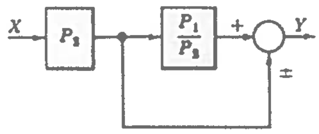
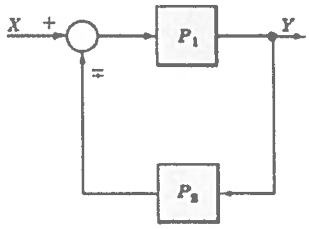
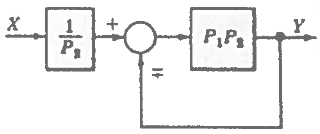
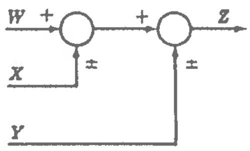
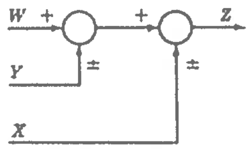
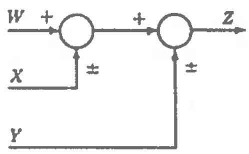
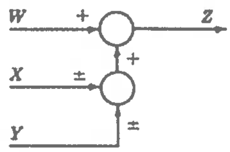
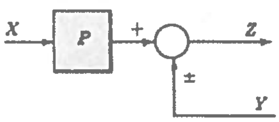
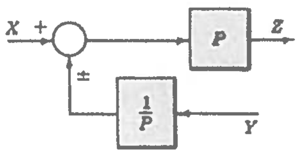
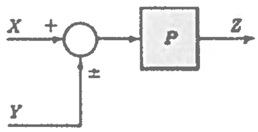
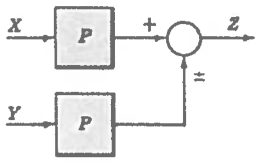
Note that in simplifying a block diagram, the product of the transfer functions in the feedforward direction must remain the same, and the product of the transfer function around the loop must remain the same.

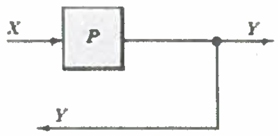
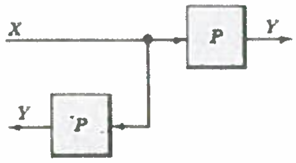
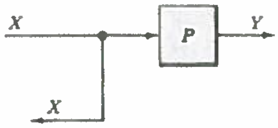
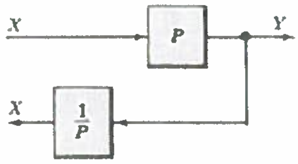
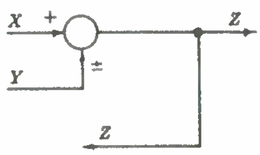
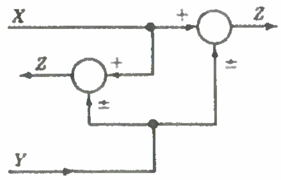
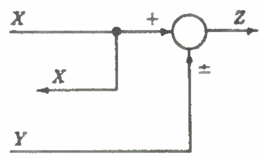
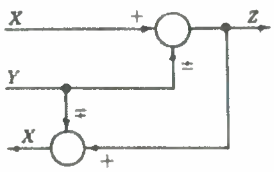
Following tables show some rules and properties must be considered during the simplification.

Block Diagram Transformations

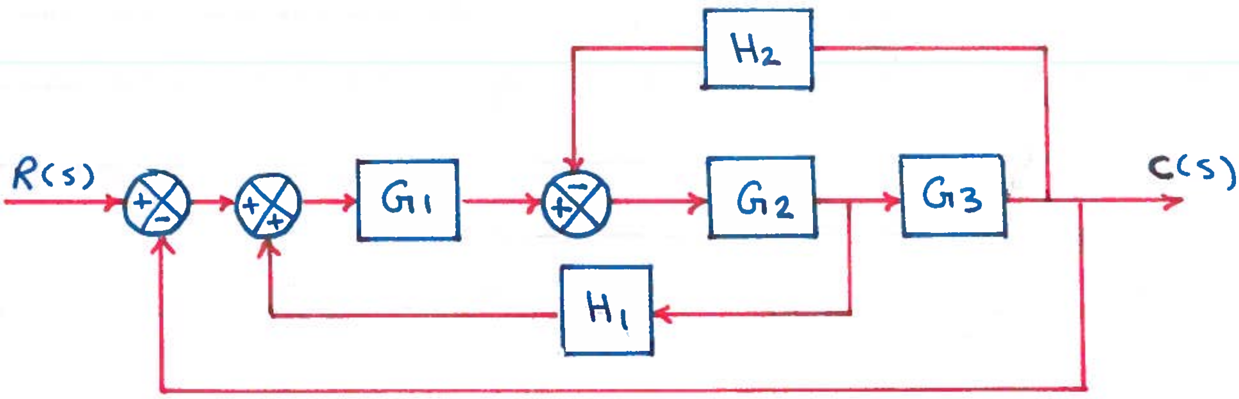
Transformation	Original Diagram	Equivalent Diagram
1. Combining blocks in cascade		
2. Moving a summing point behind a block		
3. Moving a pickoff point ahead of a block		
4. Moving a pickoff point behind a block		
5. Moving a summing point ahead of a block		
6. Eliminating a feedback loop		

BLOCK DIAGRAM ALGEBRA AND TRANSFER FUNCTIONS OF SYSTEMS

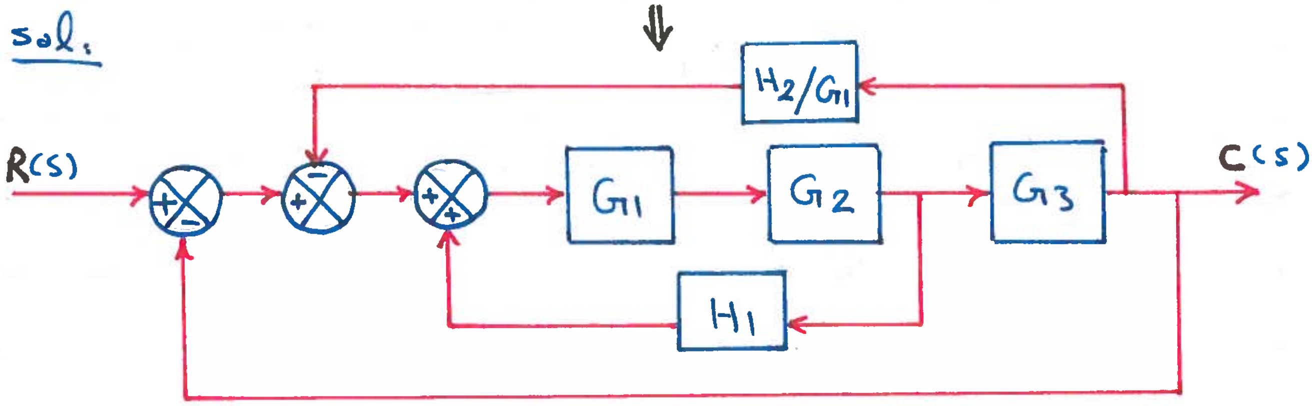
	Transformation	Equation	Block Diagram	Equivalent Block Diagram
1	Combining Blocks in Cascade	$Y = (P_1P_2)X$		
2	Combining Blocks in Parallel; or Eliminating a Forward Loop	$Y = P_1X \pm P_2X$		
3	Removing a Block from a Forward Path	$Y = P_1X \pm P_2X$		
4	Eliminating a Feedback Loop	$Y = P_1(X \mp P_2Y)$		
5	Removing a Block from a Feedback Loop	$Y = P_1(X \mp P_2Y)$		
6a	Rearranging Summing Points	$Z = W \pm X \pm Y$		
6b	Rearranging Summing Points	$Z = W \pm X \pm Y$		
7	Moving a Summing Point Ahead of a Block	$Z = PX \pm Y$		
8	Moving a Summing Point Beyond a Block	$Z = P[X \pm Y]$		

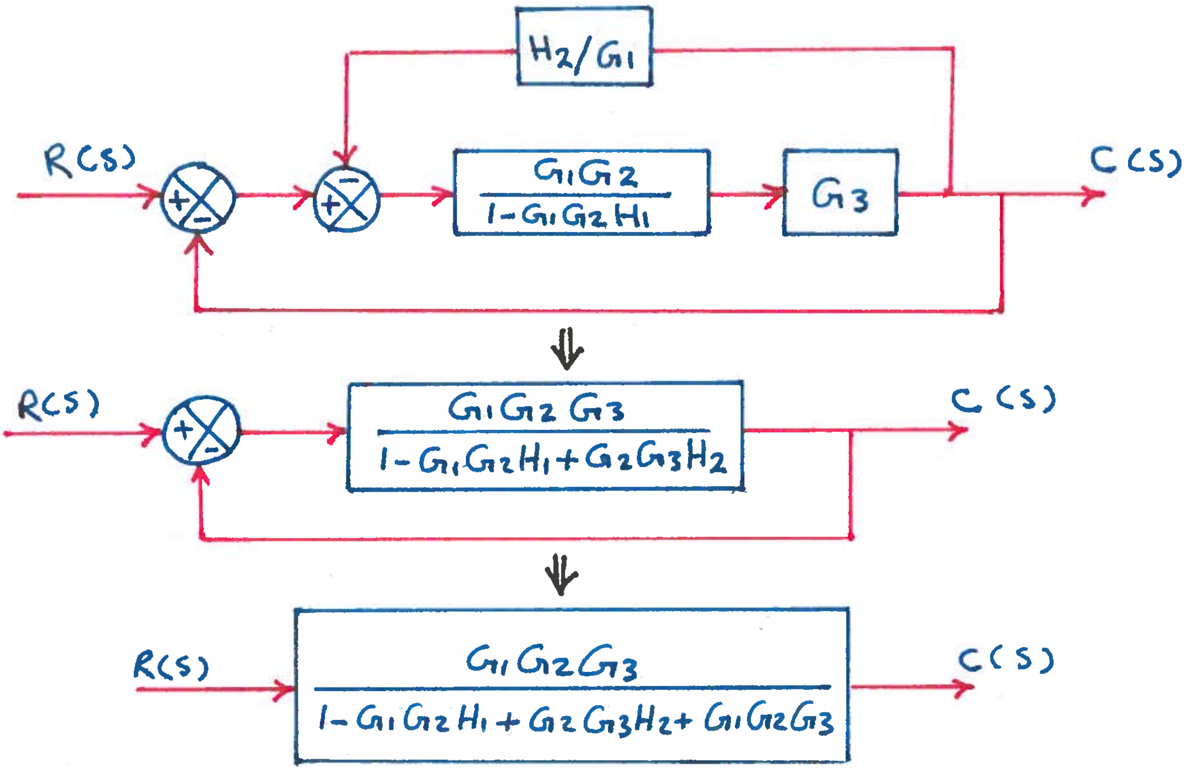
	Transformation	Equation	Block Diagram	Equivalent Block Diagram
9	Moving a Takeoff Point Ahead of a Block	$Y = PX$		
10	Moving a Takeoff Point Beyond a Block	$Y = PX$		
11	Moving a Takeoff Point Ahead of a Summing Point	$Z = X \pm Y$		
12	Moving a Takeoff Point Beyond a Summing Point	$Z = X \pm Y$		

example: Simplify the below diagram and obtain the transfer function,



sol.





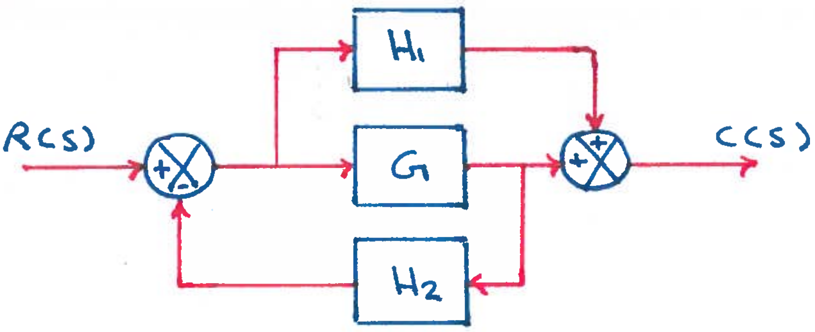
Note that the numerator of the closed_loop transfer function $C(s)/R(s)$ is the product of the transfer functions of the feed forward path, and the denominator of $C(s)/R(s)$ is equal to:

$$1-\sum(\text{product of the T.F. around each loop})$$

$$= 1-(G_1G_2H_1 - G_2G_3H_2 - G_1G_2G_3)$$

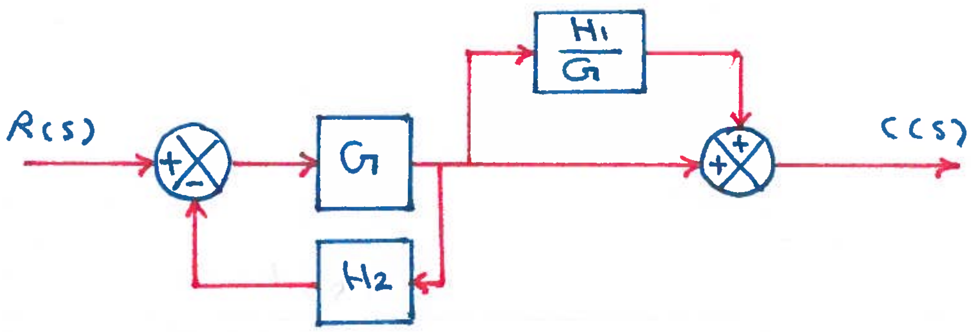
$$= 1 - G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3$$

example: Simplify the block diagram shown in figure below,

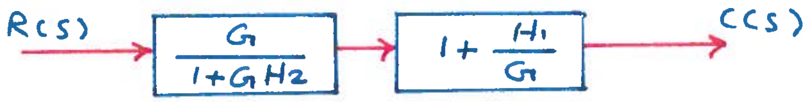


sol

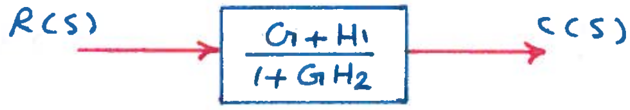
(a)



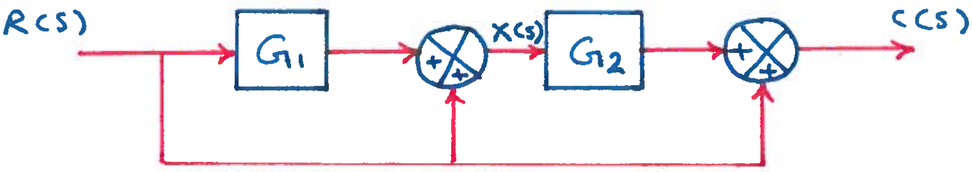
(b)



(c)

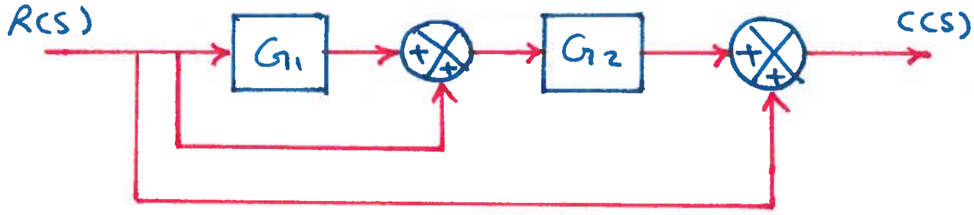


example: Simplify the block diagram shown in Figure below and obtain the transfer function relating C(s) and R(s).

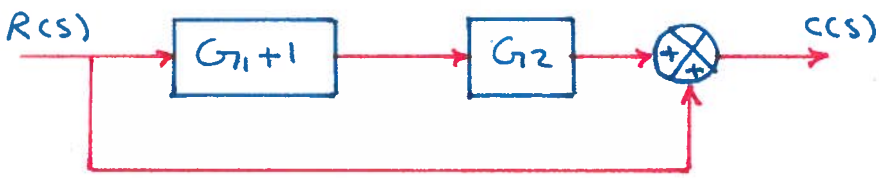


Sol

(a)



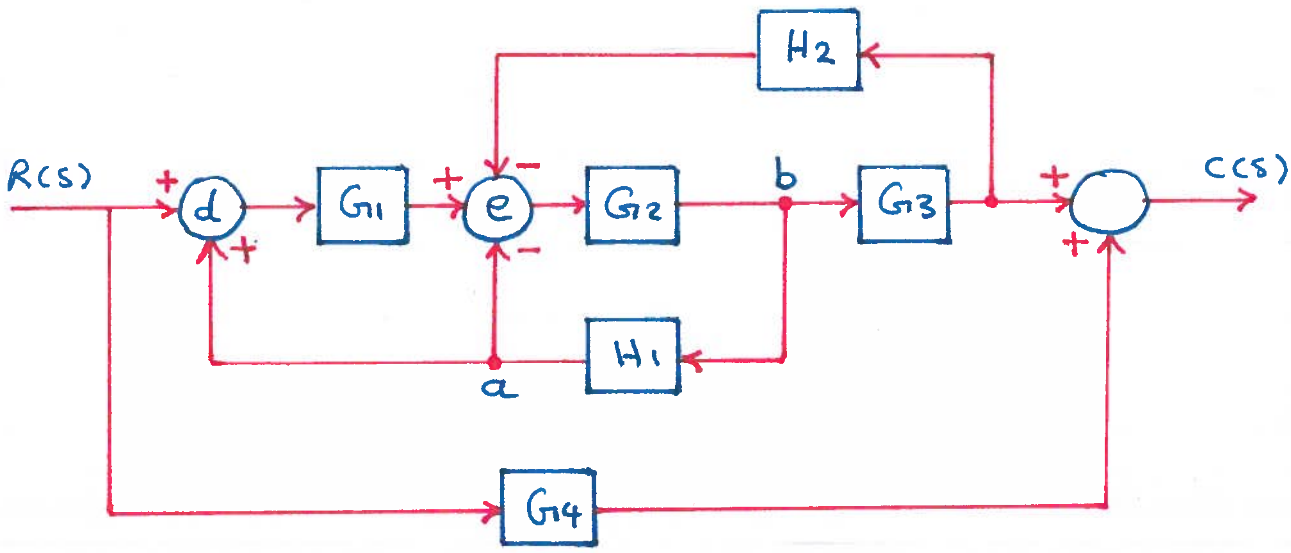
(b)



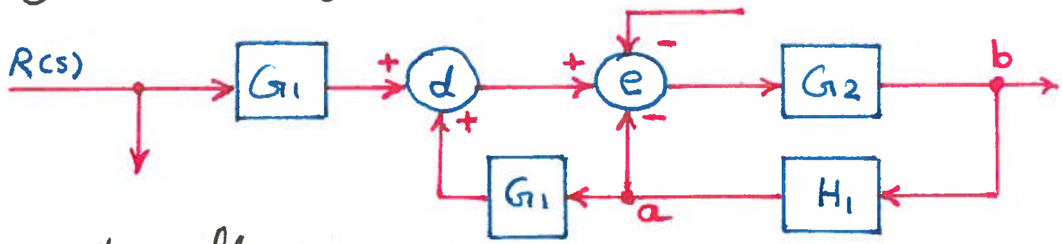
(c)

$$\rightarrow \frac{C(s)}{R(s)} = G_1 G_2 + G_2 + 1$$

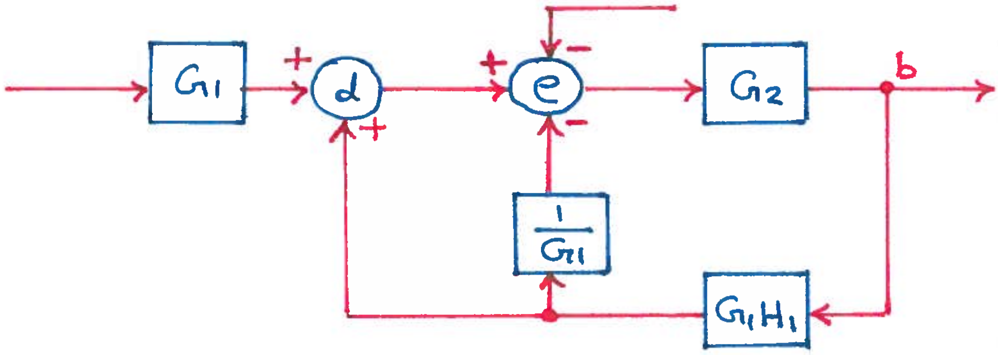
example: Reduce the block diagram given below to open loop form.



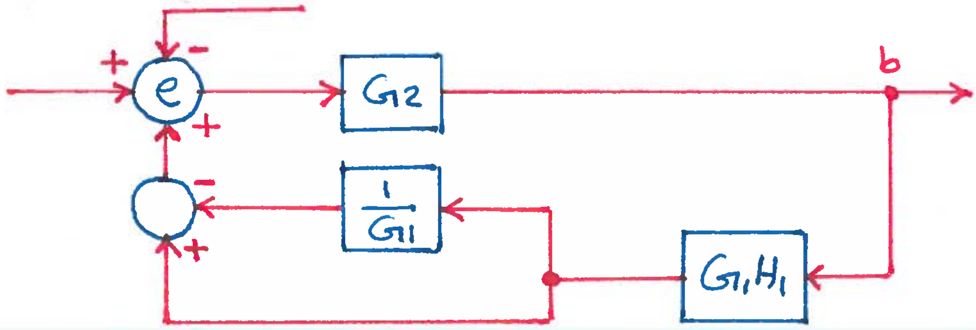
First, moving the summing point (d) beyond G_1, we get:

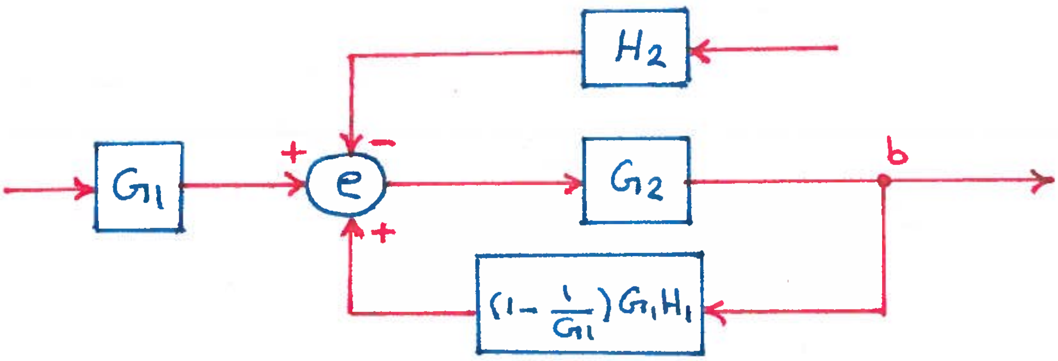


next, moving take off point (a) beyond G_1, we get:

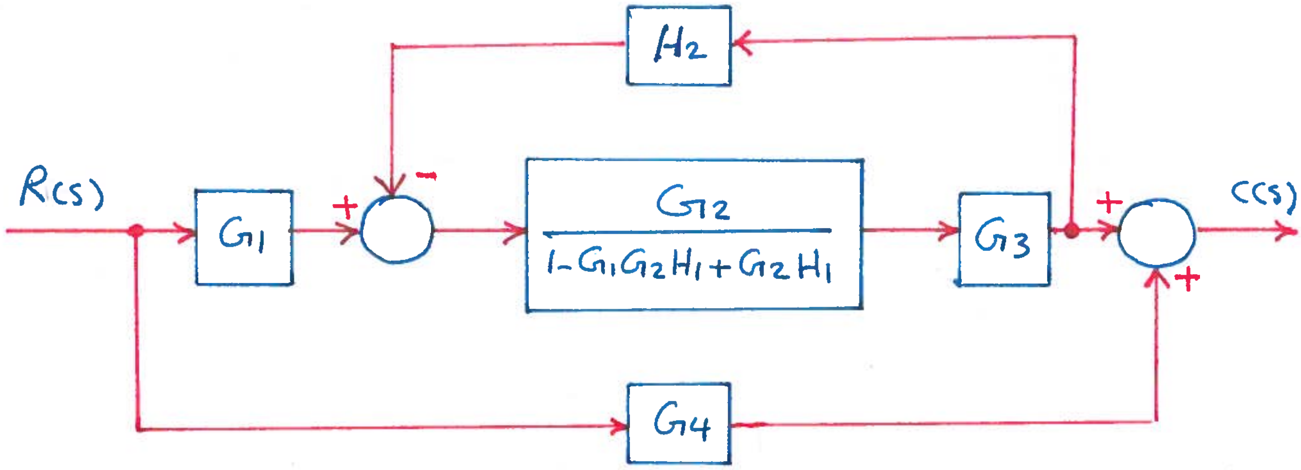


using Transformation (6b) and (2), from the tables, to combine the two lower feedback loops (from $G_1 H_1$) entering (d) and (e), we get:

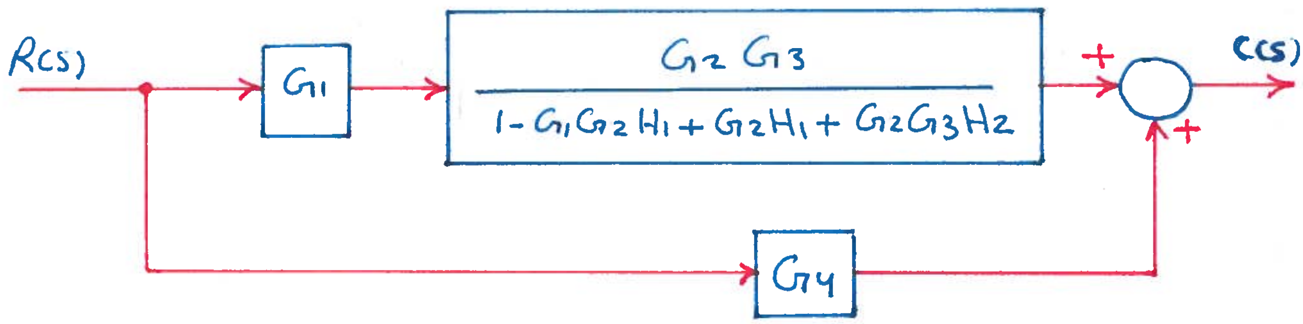




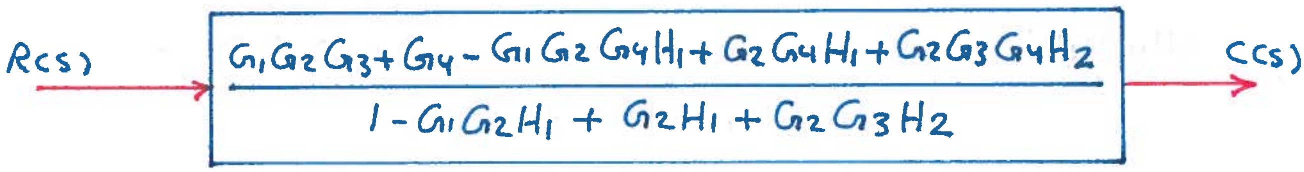
now, applying Transformation (4), from tables, to this inner loop, the system becomes:



Again, applying Transformation (4), from the tables, to the remaining feedback loop yields :



Finally, Transformation (1) and (2) from the tables, give the open-loop block diagram:



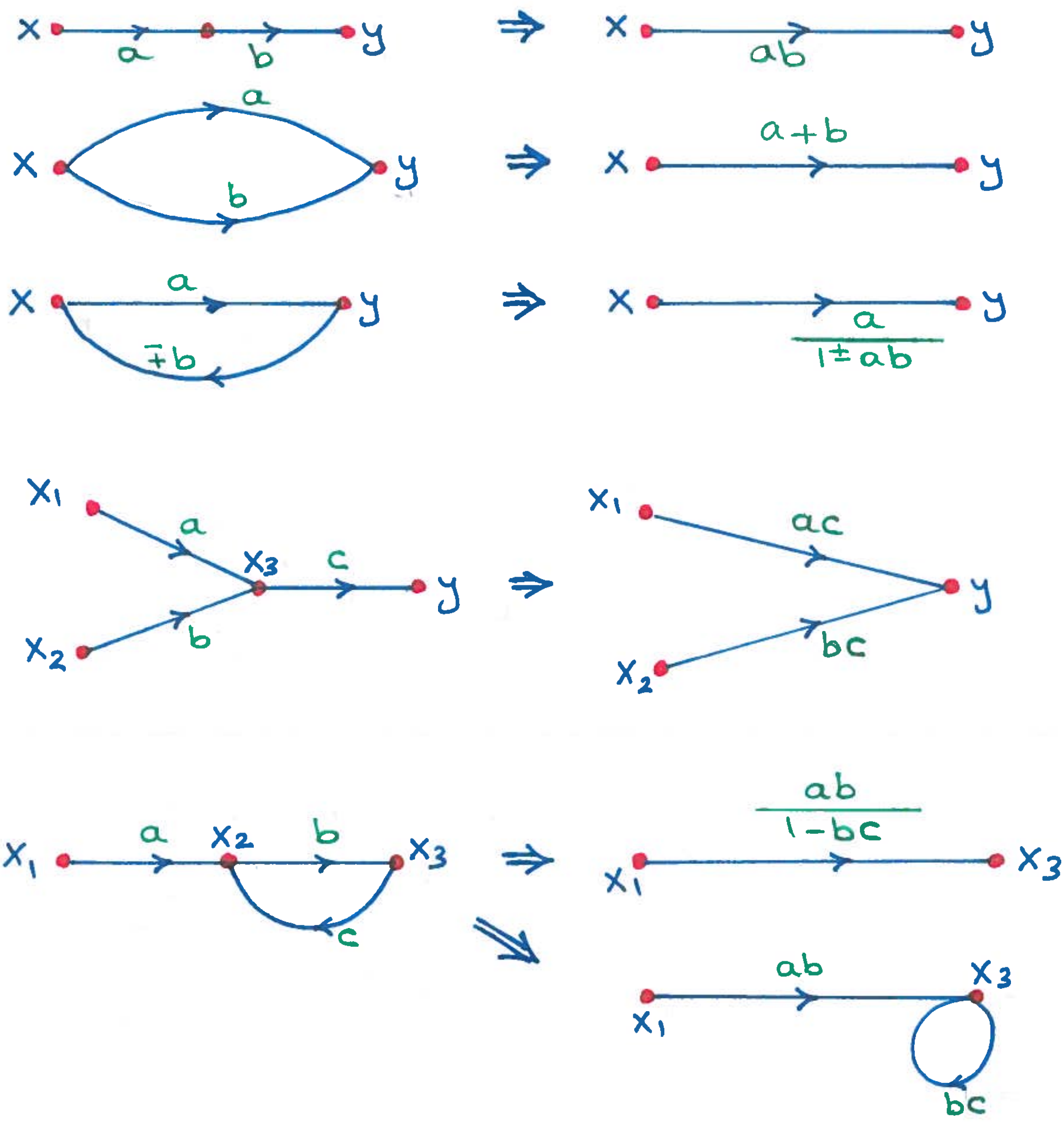
Signal Flow Graph and Mason's rule

The block diagram reduction technique is tedious and time consuming. Signal flow method gives an alternate approach for finding out transfer function of a control system. Signal flow graph is a network diagram consisting of nodes, branches, and arrows. Nodes represent variables or signals in a system. The nodes are connected by branches and arrows that indicate the direction of flow of signal.

If $y = ax$, then the signal flow graph for the equation is :

The transmittance or gain (a) is written on top of the arrow.

The following rules apply, when reducing a signal flow graph:

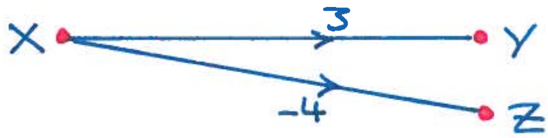


For example, Ohms low states that $V = IR$, where V is a voltage, and I a current, and R a resistance. The signal flow graph for this equation is given as:

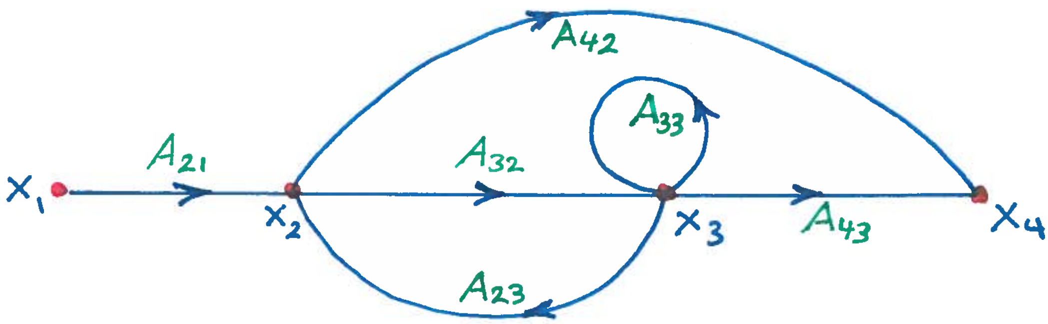
example: Represent the following equations in a signal flow graph,

$y = 3x$, $z = -4x$

Sol



Now, let's consider the below example of a signal flow graph, to define some terms,



Path: A path is a continuous unidirectional succession of branches along which no node is passed more than once. refering to the graph: x_1 to x_2 to x_3 to x_4 , and x_2 to x_3 and back to x_2 , and x_1 to x_2 to x_4.

input node/source: is a node with only outgoing branches, For example, x_1

output node/Sink: is a node with only incoming branches. For example , x_4

Forward path: is a path from input node to the output node. For example, x_1 to x_2 to x_3 to x_4 , x_1 to x_2 to x_4 are Forward paths.

Feedback path/Feedback loop: is a path which originates and terminates on the same node. For example, x_2 to x_3, back to x_2 is a feedback path.

Self-loop: is a feedback loop consisting of a single branch. For example, A_{33} is a self loop.

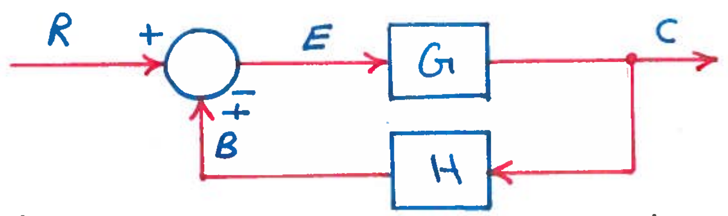
gain: is the transmission function of that branch when the transmission function is a multiplicative operator. For example, A_{33} is the gain of the self loop if A_{33} is a constant or transfer function.

path gain: is the product of the branch gains encountered in in transversing a path. For example, the path gain of the path from X_1 to X_2 to X_3 to X_4 is $A_{21}A_{32}A_{43}$.

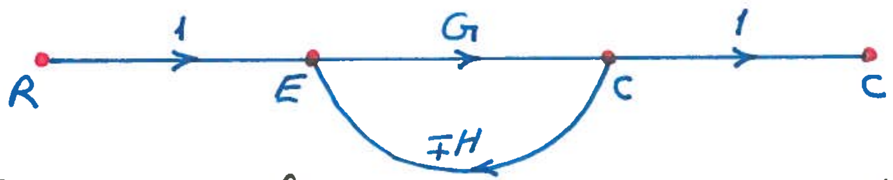
Loop gain: is the product of the branch gains of the loop. For example, the loop gain of the feedback loop from X_2 to X_3 and back to X_2 is $A_{32}A_{23}$.

Constructing the Signal flow Graph:

Consider the below block diagram,



The signal flow graph is easily constructed as shown, note that the (−) or (+) sign of the summing point is associated with H.



In general, the signal flow graph can be constructed as following:

1- write the system equations in the form,

$$X_1 = A_{11}X_1 + A_{12}X_2 + \cdots + A_{1n}X_n$$

$$X_2 = A_{21}X_1 + A_{22}X_2 + \cdots + A_{2n}X_n$$

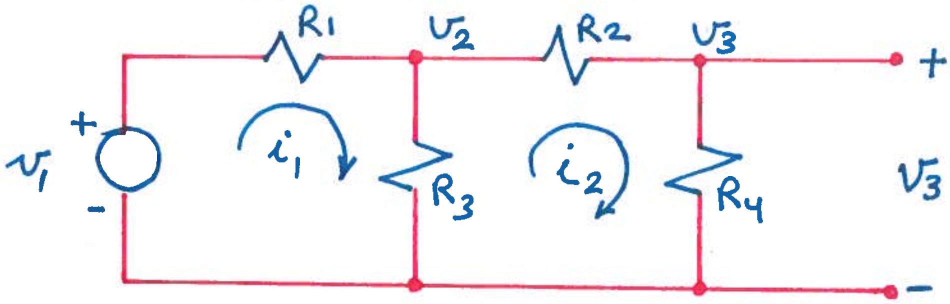
$$\vdots$$

2- arrange the nodes from left to right.

3- Connect the nodes by appropriate branches A_{11}, A_{12}, etc.

4- If the desired output node has outgoing branches, add a dummy node and a unity gain branch.

example: Consider the below circuit,



We can write four independent equations from Kirchhoff's voltage and current laws, we shall proceed from left to right,

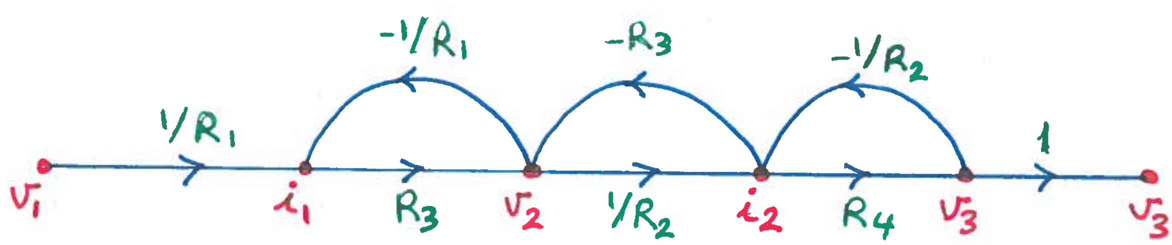
$$i_1 = \left(\frac{1}{R_1}\right) v_1 - \left(\frac{1}{R_1}\right) v_2$$

$$v_2 = R_3\, i_1 - R_3\, i_2$$

$$i_2 = \left(\frac{1}{R_2}\right) v_2 - \left(\frac{1}{R_2}\right) v_3$$

$$v_3 = R_4\, i_2$$

We have five variables (nodes); v_1, v_2, v_3, i_1, and i_2. laying out these five nodes in the same order, from left to right, with v_1 as an input node, and connecting nodes with the appropriate branches, we get the below graph. If we wish to consider v_3 as an output node, we must add a unity gain branch and another node,



Mason's Gain Formula:

A complicated block diagram can be reduced as

$$\frac{C}{R} = \frac{G}{1 \mp GH}$$

It is possible to simplify signal flow graphs in a manner similar to that of block diagram reduction.

Mason gave us a formula relating the output and input. The formula is:

$$T = \frac{C}{R} = \frac{1}{\Delta} \sum_i P_i \Delta_i$$

where T: overall gain of the system

P_i: gain of i^{th} forward path

Δ: 1 − (sum of all individual loops) + (sum of the gain product of all possible combination of two nontouching loops) − (sum of gain product of all possible combination of three nontouching loops) + (....) − (....) + ...

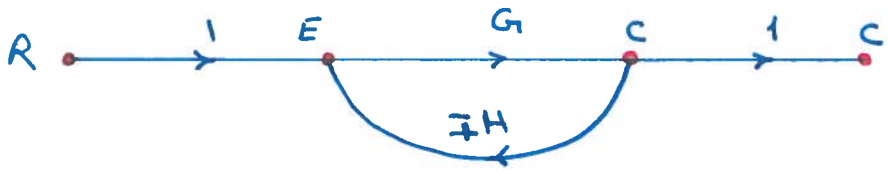
$$\rightarrow \Delta = 1 - \sum_j P_{j1} + \sum_j P_{j2} - \sum_j P_{j3} + \cdots$$

P_{jK} = j^{th} possible product of K nontouching loops

Δ_i: Same as Δ but formed by loops not touching the i^{th} forward path.

Δ is called the signal flow graph determinant or characteristic function, and if $\Delta = 0$ it is called the characteristic equation.

example: Consider the below signal flow graph,



There is only one forward path,

$$P_1 = G \quad ; \quad P_2 = P_3 = \cdots = 0$$

There is only one feedback loop, hence

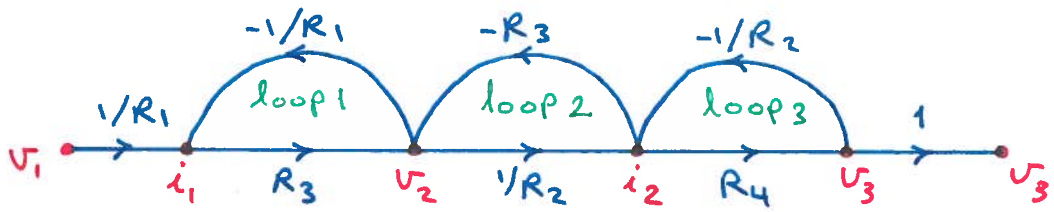
$$P_{11} = \mp GH \quad ; \quad P_{jk} = 0 \quad j \neq 1 \text{ and } k \neq 1$$

$$\rightarrow \Delta = 1 - P_{11} = 1 \pm GH$$

$$\Delta_1 = 1 - 0 = 1$$

$$\rightarrow T = \frac{C}{R} = \frac{P_1 \Delta_1}{\Delta} = \frac{G}{1 \pm GH}$$

example: Find the transfer function from the below signal flow graph,



sol There is one Forward path,

$$P_1 = \frac{1}{R_1} * R_3 * \frac{1}{R_2} * R_4 * 1 = \frac{R_3 R_4}{R_1 R_2}$$

There are three feedback loops, loop gains are,

$$P_{11} = -\frac{1}{R_1} R_3 \quad ; \quad P_{21} = -\frac{1}{R_2} R_3 \quad ; \quad P_{31} = -\frac{1}{R_2} R_4$$

There are two non touching loops, loop 1 and loop 3,

P_{12} = gain product of the only two nontouching loops $= P_{11} \cdot P_{31}$

$$\rightarrow P_{12} = \frac{R_3 R_4}{R_1 R_2}$$

There are no three loops that do not touch, therefore

$$\Delta = 1 - (P_{11} + P_{21} + P_{31}) + P_{12} = 1 + \frac{R_3}{R_1} + \frac{R_3}{R_2} + \frac{R_4}{R_2} + \frac{R_3 R_4}{R_1 R_2}$$

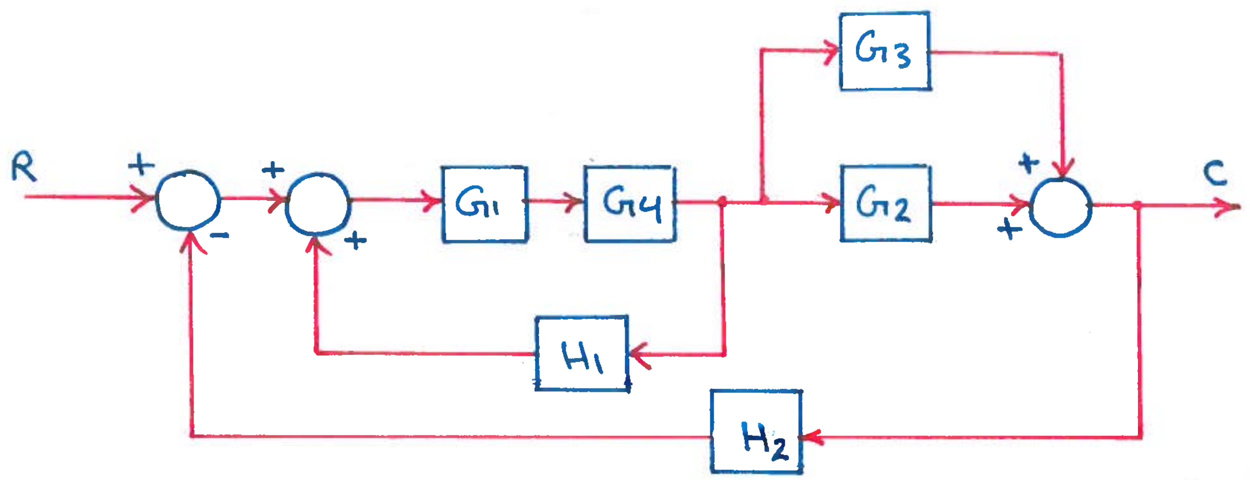
$$= \frac{R_1 R_2 + R_1 R_3 + R_1 R_4 + R_2 R_3 + R_3 R_4}{R_1 R_2}$$

Since all loops touch the forward path,

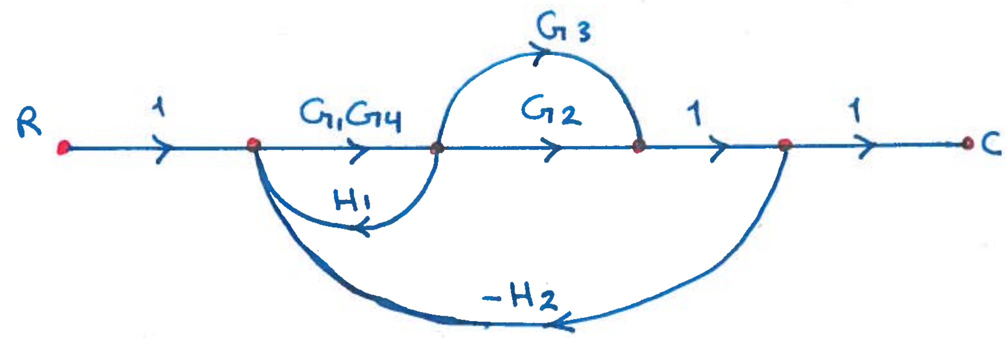
$$\rightarrow \Delta_1 = 1$$

finally $$\frac{V_3}{V_1} = \frac{P_1 \Delta_1}{\Delta} = \frac{R_3 R_4}{R_1 R_2 + R_1 R_3 + R_1 R_4 + R_2 R_3 + R_3 R_4}$$

example: Determine the control ratio C/R using Mason's rules for the below Block diagram,



Sol signal flow graph is as below:



There are three feedback loops:

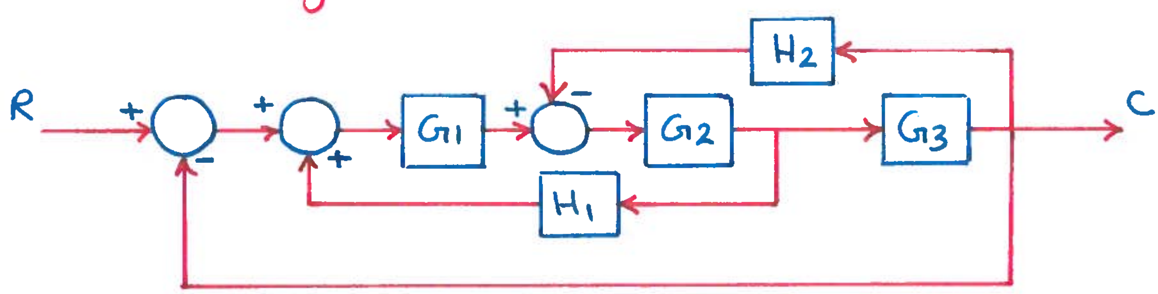
$$P_{11} = G_1 G_4 H_1 \; ; \; P_{21} = -G_1 G_2 G_4 H_2 \; ; \; P_{31} = -G_1 G_3 G_4 H_2$$

There are no nontouching loops and all loops touch both forward paths, then:

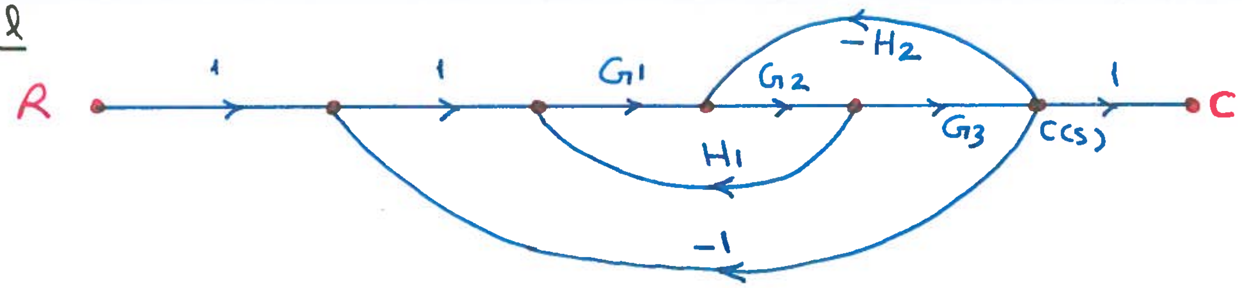
$$\Delta_1 = 1 \; , \; \Delta_2 = 1$$

$$\rightarrow T = \frac{C}{R} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta} = \frac{G_1 G_2 G_4 + G_1 G_3 G_4}{1 - G_1 G_4 H_1 + G_1 G_2 G_4 H_2 + G_1 G_3 G_4 H_2}$$

example: obtain the closed loop T.F, C/R, using Mason's rules for the block diagram shown,



sol



Forward path gain is,

$$P_1 = G_1 G_2 G_3$$

There are three individual loops,

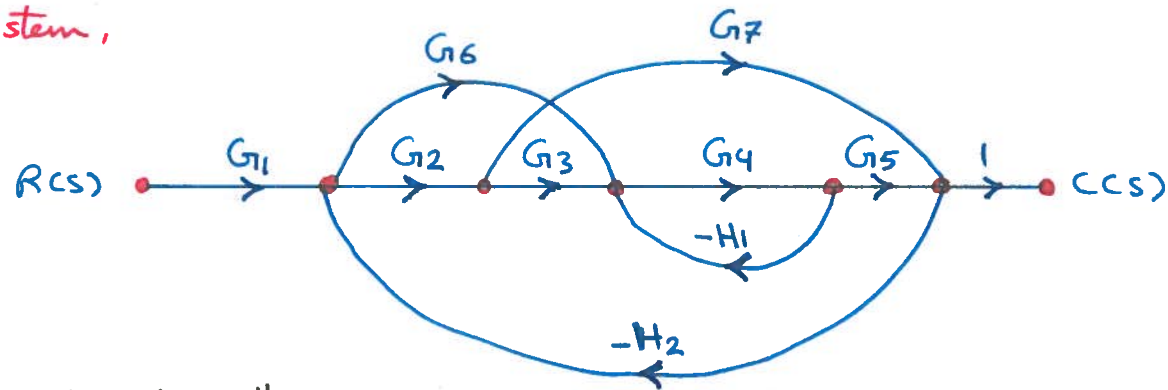
$$P_{11} = G_1 G_2 H_1 \, ; \; P_{21} = -G_2 G_3 H_2 \, ; \; P_{31} = -G_1 G_2 G_3$$

there are no nontouching loops, $\Delta_1 = 1$

$$\rightarrow \Delta = 1 - (P_{11} + P_{21} + P_{31}) = 1 - G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3$$

$$\rightarrow \frac{C}{R} = \frac{P_1 \Delta_1}{\Delta} = \frac{G_1 G_2 G_3}{1 - G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3}$$

example: obtain the closed-loop transfer function for the below system,



Forward gain paths are :

$$P_1 = G_1 G_2 G_3 G_4 G_5 \, ; \; P_2 = G_1 G_6 G_4 G_5 \, ; \; P_3 = G_1 G_2 G_7$$

There are four loops, these are :

$$P_{11} = -G_4 H_1 \, ; \; P_{21} = -G_2 G_7 H_2 \, ; \; P_{31} = -G_6 G_4 G_5 H_2 \, ;$$

$$P_{41} = -G_2 G_3 G_4 G_5 H_2$$

There are two non touching loops, P_{11} and P_{21}

$$\rightarrow \Delta = 1 - (P_{11} + P_{21} + P_{31} + P_{41}) + P_{11} P_{21} + 0$$

$$\rightarrow \Delta = 1 + G_4 H_1 + G_2 G_7 H_2 + G_6 G_4 G_5 H_2 + G_2 G_3 G_4 G_5 H_2 + G_2 G_4 G_7 H_1 H_2$$

now, we have all the loops touching P_1, $\rightarrow \Delta_1 = 1$

for P_2, all loops are touching it as well;

therefore $\Delta_2 = 1$

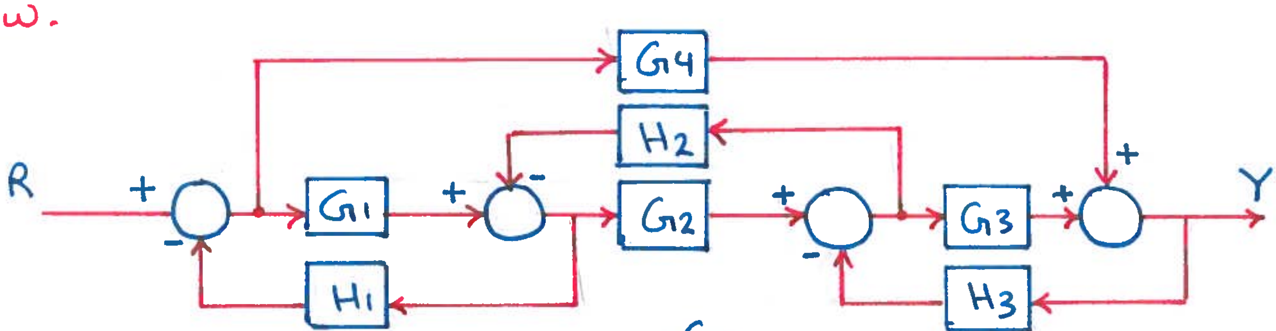
for P_3, we have nontouching loop for P_3 which is P_{11},

$\rightarrow \Delta_3 = 1 - P_{11} = 1 + G_4 H_1$

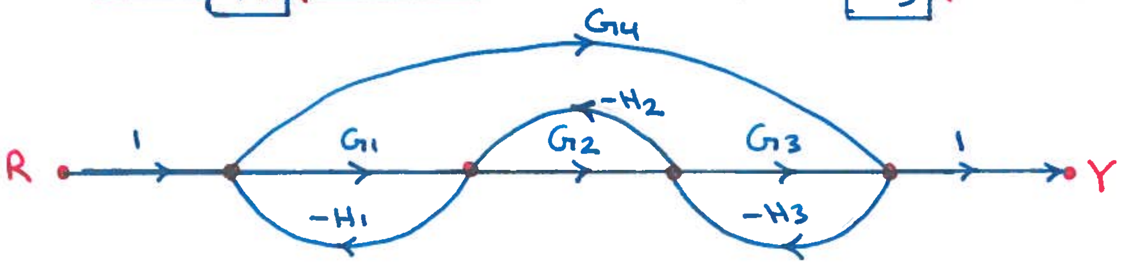
$\rightarrow T.F = \frac{C}{R} = \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3)$

$$= \frac{G_1G_2G_3G_4G_5 + G_1G_6G_4G_5 + G_1G_2G_7(1+G_4H_1)}{1+G_4H_1+G_2G_7H_2+G_6G_4G_5H_2+G_2G_3G_4G_5H_2+G_2G_4G_7H_1H_2}$$

example: Consider the block diagram shown below, draw the equivalent signal flow graph and find the transfer function using Mason's law.



sol



There are two forward Paths, path gains are:

$P_1 = G_1G_2G_3$; $P_2 = G_4$

There are Four individual loops with loop gains:

$P_{11} = -G_1H_1$; $P_{21} = -G_2H_2$; $P_{31} = -G_3H_3$; $P_{41} = -G_4H_1H_2H_3$

There is one combination of two nontouching loops, their loop gain product is : $P_{11} \cdot P_{31} = G_1G_3H_1H_3$

$\rightarrow \Delta = 1 - (-G_1H_1 - G_2H_2 - G_3H_3 - G_4H_1H_2H_3) + G_1G_3H_1H_3 + 0$

now, we have all loops touching $P_1 \rightarrow \Delta_1 = 1$

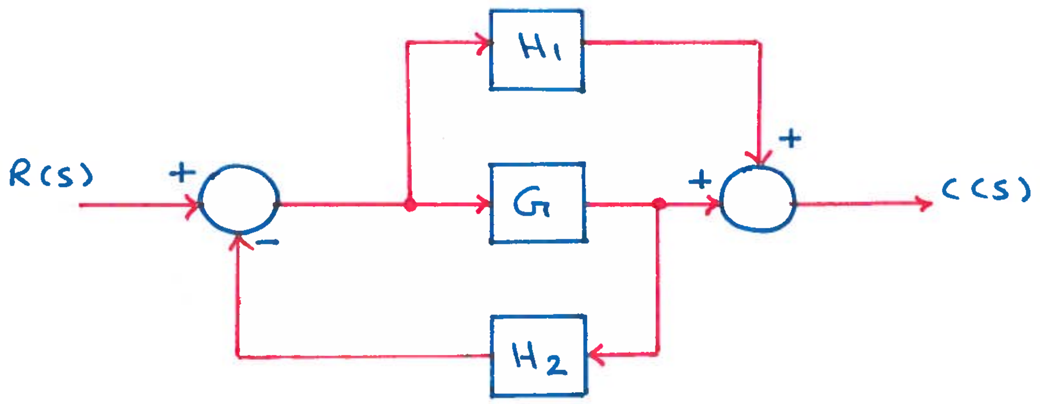
We have one loop (P_{21}) touching P_2

$\rightarrow \Delta_2 = 1 - (P_{21}) = 1 - (-G_2H_2) = 1 + G_2H_2$

$$\rightarrow T = \frac{Y}{R} = \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2)$$

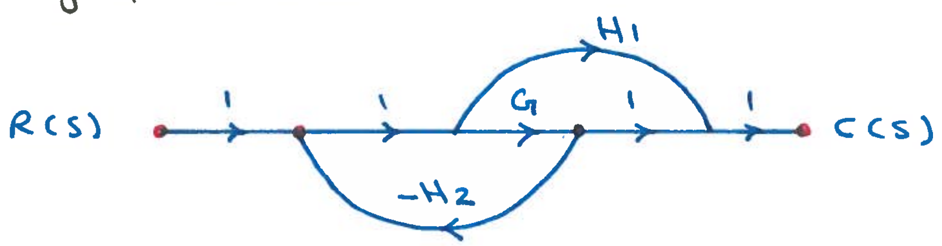
$$= \frac{G_1G_2G_3 + G_4(1 + G_2H_2)}{1 + G_1H_1 + G_2H_2 + G_3H_3 + G_4H_1H_2H_3 + G_1G_3H_1H_3}$$

example: use Mason's rules to obtain the transfer function for the below block diagram,



Sol

Signal flow graph as below,



We have two forward paths with gains:

$P_1 = G$; $P_2 = H_1$

We have one loop:

$P_{11} = -GH_2$

this loop is touching P_1 and P_2

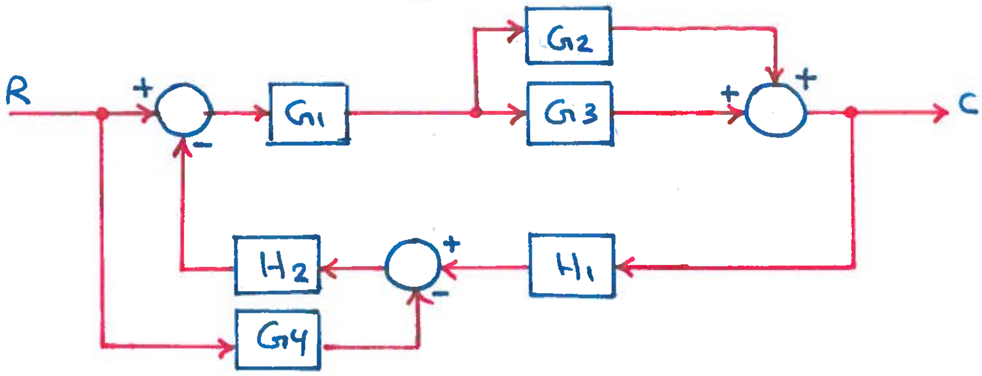
hence $\Delta_1 = \Delta_2 = 1$

$\Delta = 1 - (P_{11}) + 0 = 1 + GH_2$

$$\rightarrow T.F = \frac{C}{R} = \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2)$$

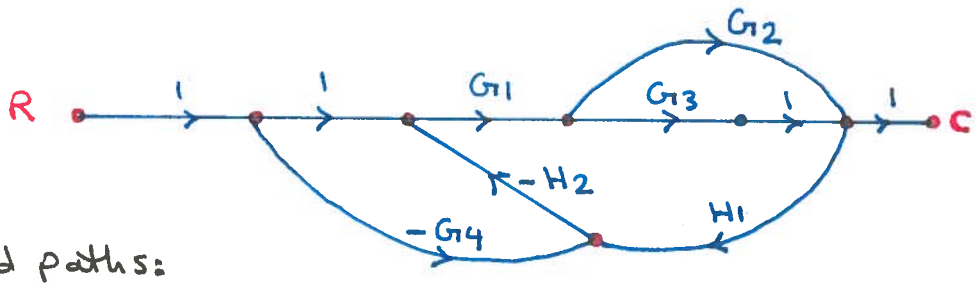
$$= \frac{G + H_1}{1 + GH_2}$$

example: Find C/R for the system whose block diagram representation is shown, by signal flow graph technique.



Sol

The signal flow graph will be as shown below:



Forward paths:

$P_1 = G_1 G_3$; $P_2 = G_1 G_2$; $P_3 = G_1 G_3 G_4 H_2$; $P_4 = G_1 G_2 G_4 H_2$

Loops: $P_{11} = -G_1 G_3 H_1 H_2$; $P_{21} = -G_1 G_2 H_1 H_2$

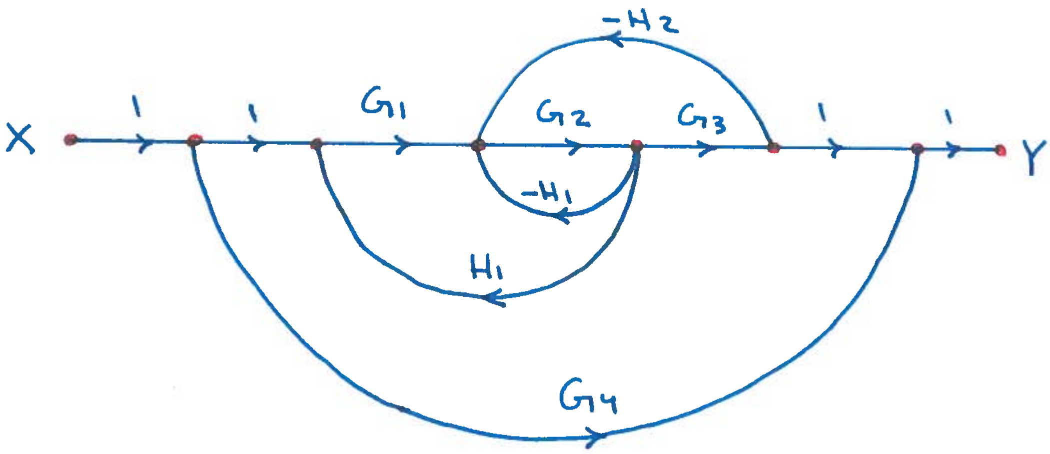
loops are touching all paths ; $\rightarrow \Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 1$

$\rightarrow \Delta = 1 - (P_{11} + P_{21}) = 1 + G_1 G_2 H_1 H_2 + G_1 G_3 H_1 H_2$

$$\rightarrow \frac{C}{R} = \frac{1}{\Delta}(P_1 \Delta_1 + P_2 \Delta_2 + P_3 \Delta_3 + P_4 \Delta_4)$$

$$= \frac{G_1 G_3 + G_1 G_2 + G_1 G_3 G_4 H_2 + G_1 G_2 G_4 H_2}{1 + G_1 G_2 H_1 H_2 + G_1 G_3 H_1 H_2}$$

example: obtain the closed loop transfer function for the system shown,



Sol

We have two forward paths:

$P_1 = G_1 G_2 G_3$

$P_2 = G_4$

We have three feedback loops :

$P_{11} = G_1 G_2 H_1$

$P_{21} = -G_2 H_1$

$P_{31} = -G_2 G_3 H_2$

$$\Delta = 1 - (P_{11} + P_{21} + P_{31})$$
$$= 1 - G_1 G_2 H_1 + G_2 G_3 H_2 + G_2 H_1$$

for P_1, there are no nontouching loops for it,

$\rightarrow \Delta_1 = 1$

for P_2, all feedback loops are non touching and since Δ_i is same as Δ but it is formed by nontouching loops

$$\rightarrow \Delta_2 = \Delta = 1 - G_1 G_2 H_1 + G_2 G_3 H_2 + G_2 H_1$$

$$\rightarrow \text{Transfer function} = \frac{Y}{X} = \frac{1}{\Delta}(P_1 \Delta_1 + P_2 \Delta_2)$$

$$\rightarrow \frac{Y}{X} = \frac{G_1 G_2 G_3 + G_4(1 + G_2 H_1 - G_1 G_2 H_1 + G_2 G_3 H_2}{1 - G_1 G_2 H_1 + G_2 G_3 H_2 + G_2 H_1}$$

Multivariable Systems and Transfer Matrices

A multivariable system is one with more than one input (multiinput, MI), more than one output (multioutput, MO), or both (multinput-multioutput, MIMO).

If we assume that a System with multiple inputs and multiple outputs, Then the transfer function that relates the Laplace Transform of the output vector to the Laplace Transform of the input vector is called The Transfer Matrix between the output vector and the input vector.

If such a system has m inputs $(u_1 \ldots\ldots u_m)$, and n outputs $(x_1 \ldots\ldots x_n)$, then

$$\begin{bmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_n(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1m}(s) \\ G_{21}(s) & G_{22}(s) & \cdots & G_{2m}(s) \\ \vdots & \vdots & & \vdots \\ G_{n1}(s) & G_{n2}(s) & \cdots & G_{nm}(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \\ \vdots \\ U_m(s) \end{bmatrix}$$

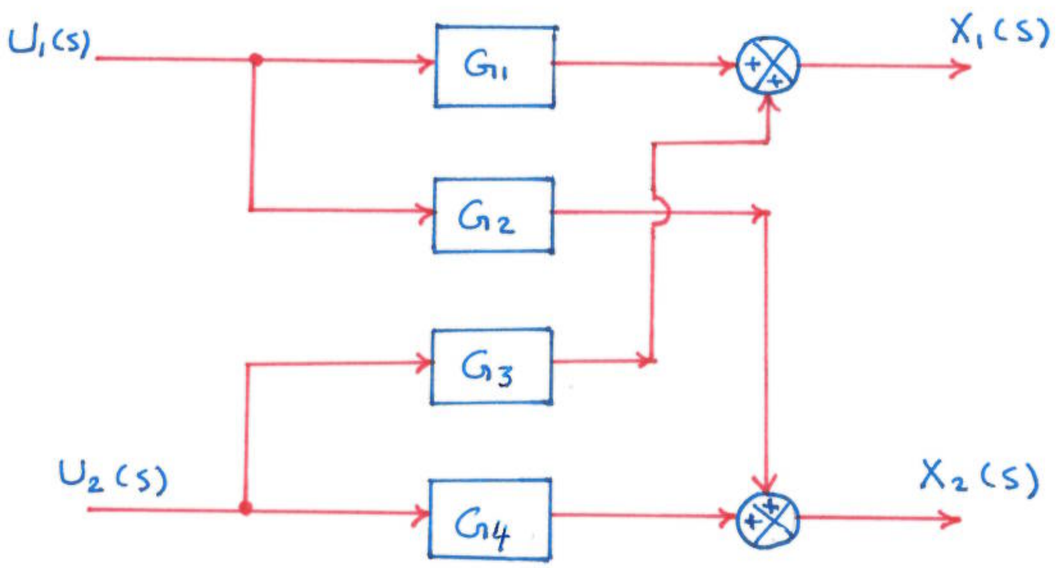
or $\quad X(s) = G(s)\,U(s)$

$X(s)$ is the Laplace Transform of the output vector,

$U(s)$ is the Laplace Transform of the input vector,

$G(s)$ is The Transfer matrix.

For example, let's Consider the below system,



The relations between inputs and outputs can be written as,

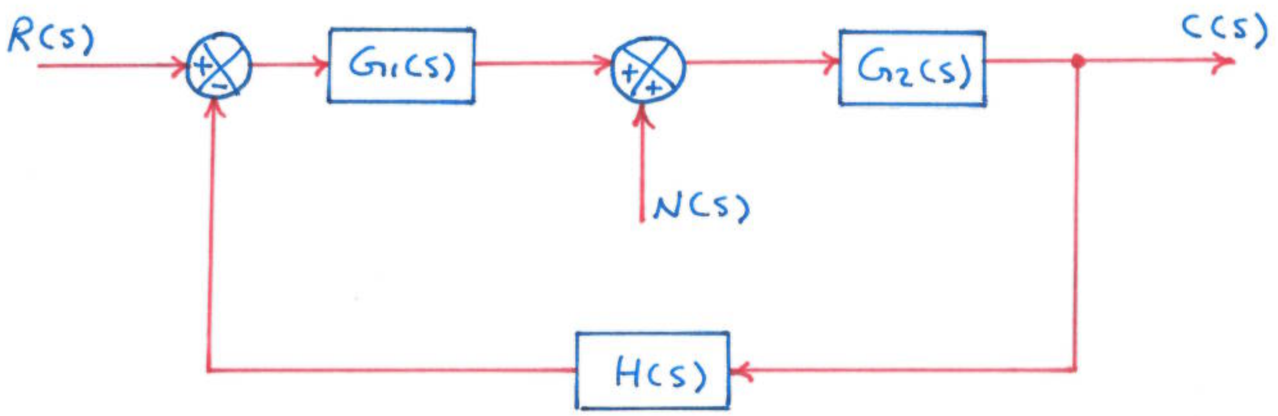
$X_1(s) = G_1(s)U_1(s) + G_3(s)U_2(s)$

$X_2(s) = G_2(s)U_1(s) + G_4(s)U_2(s)$

So we can re-write it in a matrix form (vector matrix),

$$\Rightarrow \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} G_1(s) & G_3(s) \\ G_2(s) & G_4(s) \end{bmatrix} \cdot \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

example: obtain the Transfer matrix between the output and the input for the system Given,



Sol

The output C(s) is:

$$C(s) = \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H(s)} R(s) + \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)} N(s)$$

we can write it in vector matrix form to be:

$$C(s) = \begin{bmatrix} \dfrac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H(s)} & \dfrac{G_2(s)}{1+G_1(s)G_2(s)H(s)} \end{bmatrix} \cdot \begin{bmatrix} R(s) \\ N(s) \end{bmatrix}$$

State Space Representation

A modern complex system may have many inputs and many outputs, and these may be interrelated in a complicated manner. To analyze such a system, it is essential to reduce the complexity of the mathematical expressions. The state space approach to system analysis is the best suited from this viewpoint.

While conventional control theory is based on the input-output relationship, or transfer function, modern control theory is based on the description of system equations in terms of n first-order differential equations, which may be combined into a first-order vector matrix differential equation that simplifies the mathematical representation of systems equations.

The state space representation is given as:

$$\dot{x} = \mathbf{A}x + \mathbf{B}u \dashrightarrow \text{The state equation}$$

$$y = \mathbf{C}x + \mathbf{D}u \dashrightarrow \text{The output equation}$$

The above two equations represent a Standard form that representing all linear physical systems, where:

x : The state variable,

$\dot{x}$: The derivative of x, which gives the prediction or expectation of how the state of x will be in the future,

y : The output or the Controlled Variable,

$\mathbf{A}$: The state matrix of the system, or characteristic matrix, that gives the property or characteristics of the system interms of its components.

$\mathbf{B}$: The input matrix or the parameters or gains related to the input, or scalling the input Variable,

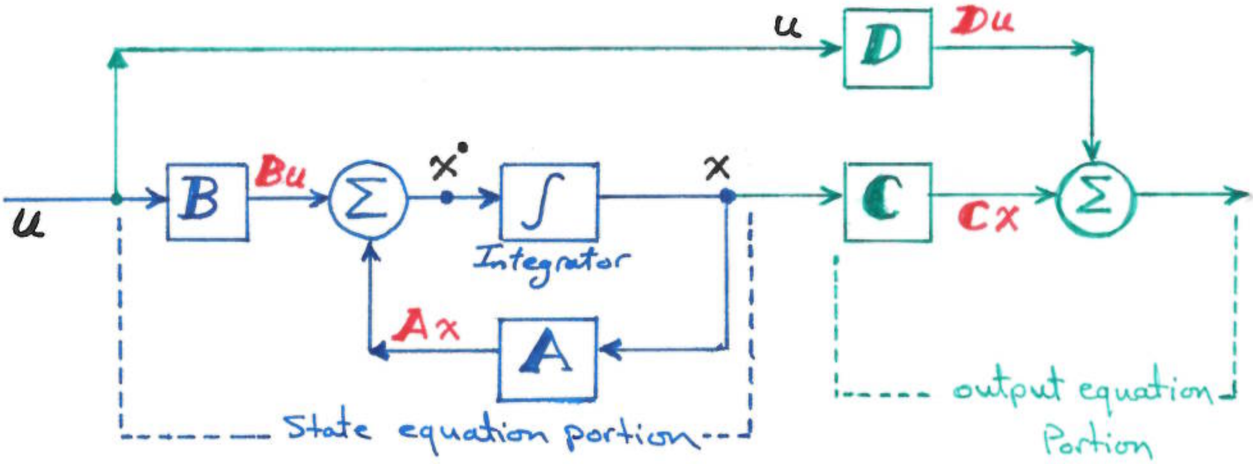
$\mathbf{C}$: scalling matrix of the output

$\mathbf{D}$: scalling matrix or feedforward matrix,

u : The input variable,

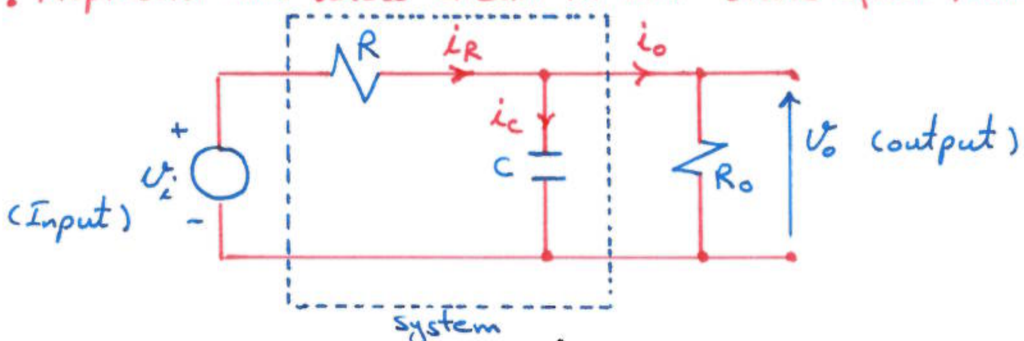
let's represent the equations in the form of a Block diagram, as shown,

$$\dot{x} = \mathbf{A}x + \mathbf{B}u$$
$$y = \mathbf{C}x + \mathbf{D}u$$



every system can be represented in above form. we have the input portion ($\mathbf{B}u$), the output portion ($\mathbf{C}x$), feed forward portion ($\mathbf{D}u$), and the integral and feedback portion which gives the characteristics of the system, that is the Characteristic matrix ($\mathbf{A}x$).

example: Represent the below circuit in the state space form,



sol

1- First step to be done is to identify the energy storing element.

Here we have the capacitor C, so we have 1 energy storing element.

- number of energy storing elements are directly indicating the Order of the system.

→ Here, we have 1st order system.

2- Identify the variables that will be used for modeling u & x.

The system response is dependent on the input of the system (u), and the integrator output values (x) which is the state variables,

here $u = v_i$ and $x = v_c$

3- Starting with dynamic elements, obtain the differential equations.

* number of differential equations = Order of the system = the number of energy storing elements (memory elements),

We have $u = v_i$, $x = v_c$, and R & C form **A**,

$$\rightarrow \quad v_c = \frac{1}{C}\int i_c \, dt$$

but $i_c = i_R - i_o = \dfrac{v_i - v_c}{R} - \dfrac{v_c}{R_o}$

$$\rightarrow \quad \frac{dv_c}{dt} = \frac{1}{RC}(v_i - v_c) - \frac{1}{R_o C} v_c$$

4- Arrange the equations into form:

$$\dot{x} = \mathbf{A}x + \mathbf{B}u$$

$$\text{and} \quad y = \mathbf{C}x + \mathbf{D}u$$

$$\rightarrow \dot{v_c} = \left(\frac{-1}{RC} - \frac{1}{R_o C}\right) v_c - \frac{1}{RC} v_i$$ which is the state equation

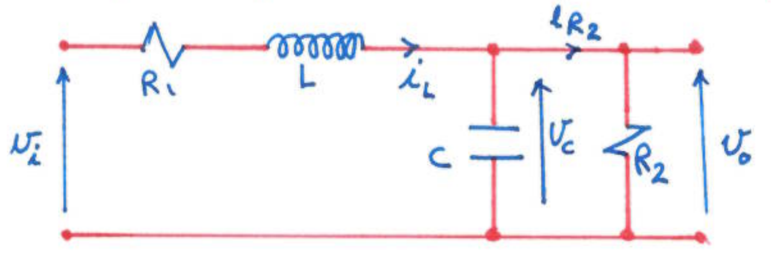
where $\mathbf{A} = \left(\frac{-1}{RC} - \frac{1}{R_o C}\right)$ & $\mathbf{B} = -\frac{1}{RC}$

also : $v_o = [1][v_c] + [0]\, v_i$ which is the output equation

where $\mathbf{C} = (1)$ and $\mathbf{D} = 1$

& $u = v_i$, $x = v_c$, $\dot{x} = \dfrac{dv_c}{dt} = \dot{v_c}$

example: write the state space equations for the following circuit,



Sol

number of energy storage elements is 2, C & L
where C : stores $\frac{1}{2} C v_c^2$
and L : stores $\frac{1}{2} L i_L^2$
→ we have 2^{nd} order system, and 2 differential equations,
variables :
input variable $u = [v_i]$, output is $v_o = v_c$
state variable $x = \begin{bmatrix} i_L \\ v_c \end{bmatrix}$ which is the state vector

Parameters of the systems : R_1, L, C, and R_2
now we write the differential equations:
Voltage across L is : $L \frac{di_L}{dt} = v_i - v_{R_1} - v_c$

$$\rightarrow L \frac{di_L}{dt} = v_i - v_{R_1} - v_c = v_i - i_L R_1 - v_c$$

$$\rightarrow \dot{i_L} = \frac{di_L}{dt} = -\frac{R_1}{L} i_L - \frac{1}{L} v_c + \frac{1}{L} v_i \quad \text{---- (1)}$$

Current through C is : $C \frac{dv_c}{dt} = i_L - i_{R_2}$
but $i_{R_2} = v_c / R_2$

$$\rightarrow C \frac{dv_c}{dt} = i_L - \frac{v_c}{R_2}$$

$$\rightarrow \frac{dv_c}{dt} = \dot{v_c} = \frac{1}{C} i_L - \frac{1}{R_2 C} v_c \quad \text{---- (2)}$$

now we will arrange equations (1) and (2) in the form of matrix :

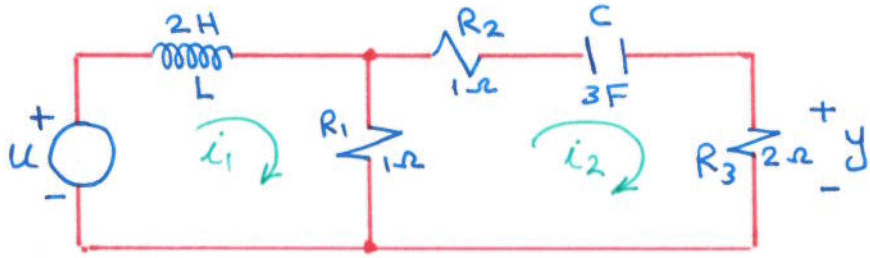
$$\underbrace{\begin{bmatrix} \dot{i_L} \\ \dot{v_c} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{R_2 C} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} i_L \\ v_c \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} v_i \end{bmatrix}}_{u}$$

and for the output :

$$\underbrace{v_o}_{y} = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \cdot \underbrace{\begin{bmatrix} i_L \\ v_c \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} v_i \end{bmatrix}}_{u}$$

note : there is no direct feedforward from input to the output, $\rightarrow D = 0$

example: For the circuit shown below, give the state space equations,



Sol

Variables are i_L, v_c

$$\rightarrow x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} i_L \\ v_c \end{bmatrix} \quad ; \text{where } x_1 = i_L \text{ and } x_2 = v_c$$

also $v_L = L \dfrac{di_L}{dt} = L \dot{i}_L = L \dot{x}_1 = 2\dot{x}_1$

and $i_c = c \dfrac{dv_c}{dt} = c\dot{v}_c = 3\dot{v}_c = 3\dot{x}_2$

now, from loop ①

$$u = v_L + v_{R_1} = v_L + R_1 (i_1 - i_2)$$

$$i_1 = i_L \quad \text{and} \quad i_2 = i_c$$

$$\rightarrow u = 2\dot{x}_1 + (i_L - i_c)$$

$$\rightarrow u = 2\dot{x}_1 + (x_1 - 3\dot{x}_2) \quad \text{---} ①$$

from loop ②

$$(i_2 - i_1) R_1 + v_c + i_2 R_2 + i_2 R_3 = 0$$

$$(i_c - i_L) R_1 + v_c + i_c R_2 + i_c R_3 = 0$$

$$(3\dot{x}_2 - x_1) + x_2 + 3\dot{x}_2 + 6\dot{x}_2 = 0$$

$$\rightarrow 12\dot{x}_2 = x_1 - x_2 \Rightarrow \dot{x}_2 = \frac{1}{12}(x_1 - x_2) \quad \text{---} ②$$

sub. (2) into (1)

$$\rightarrow u = 2\dot{x}_1 + (x_1 - 3[\tfrac{1}{12}(x_1 - x_2)])$$

$$\rightarrow 2\dot{x}_1 = -x_1 + \frac{3}{12}x_1 - \frac{3}{12}x_2 + u$$

$$\rightarrow \dot{x}_1 = -\frac{3}{8}x_1 - \frac{1}{8}x_2 + \frac{1}{2}u \quad --- (3)$$

now writing eq. (3) and (2) again:

$$\dot{x}_1 = (-\tfrac{3}{8})x_1 - (\tfrac{1}{8})x_2 + (\tfrac{1}{2})u$$

and
$$\dot{x}_2 = (\tfrac{1}{12})x_1 - (\tfrac{1}{12})x_2 + (0)u$$

Therefore, in matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{8} & -\frac{1}{8} \\ \frac{1}{12} & -\frac{1}{12} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \cdot u$$

or

$$\underbrace{\begin{bmatrix} \dot{i}_L \\ \dot{v}_c \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} -\frac{3}{8} & -\frac{1}{8} \\ \frac{1}{12} & -\frac{1}{12} \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} i_L \\ v_c \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}}_{B} \cdot u$$

for the output:

$$y = Cx + Du$$

$$v_o = i_c R_3 = 3\dot{x}_2 R_3 = 6\dot{x}_2$$

from eq. (2) we have $\dot{x}_2 = \frac{1}{12}(x_1 - x_2)$

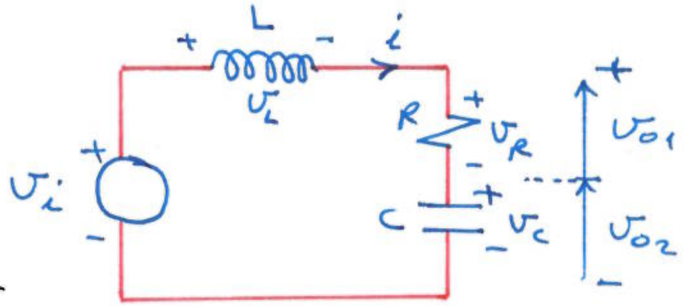
$$\rightarrow v_o = 6 * \frac{1}{12}(x_1 - x_2)$$

$$\rightarrow v_o = \frac{1}{2}x_1 - \frac{1}{2}x_2 = y$$

$\rightarrow$ in matrix form:

$$y = \underbrace{\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} u$$

example: Find the state space modeling for the circuit shown,



sol

We have 2 Storage elements

→ we have second Order System, and we have two differential equations,

note that $i_L = i_R = i_c = i$

also $v_L = L\,di/dt$, $v_R = iR$; and also $i_c = i = C\dfrac{dv_c}{dt}$

Variables i and v_c

where $i = x_1$; $v_c = x_2 \Rightarrow \dot{x}_1 = \dot{i}$ and $\dot{x}_2 = \dot{v}_c$

Parameters R, L, and C

now: from KVL, $v_i - v_L - v_R - v_c = 0$

$$\rightarrow v_i - L\dot{i} - iR - v_c = 0$$

$$\rightarrow \dot{i} = -\frac{R}{L} i - \frac{1}{L} v_c + \frac{1}{L} v_i \quad \text{----}(1)$$

we need 2^{nd} equation,

We have $\dot{v}_c = \frac{1}{C} i$, (since $i = C\dfrac{dv_c}{dt} = C\dot{v}_c$) --(2)

$$\rightarrow \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{X}} = \underbrace{\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} v_i \end{bmatrix}}_{u} \text{ ; state equation}$$

now for the output equations:

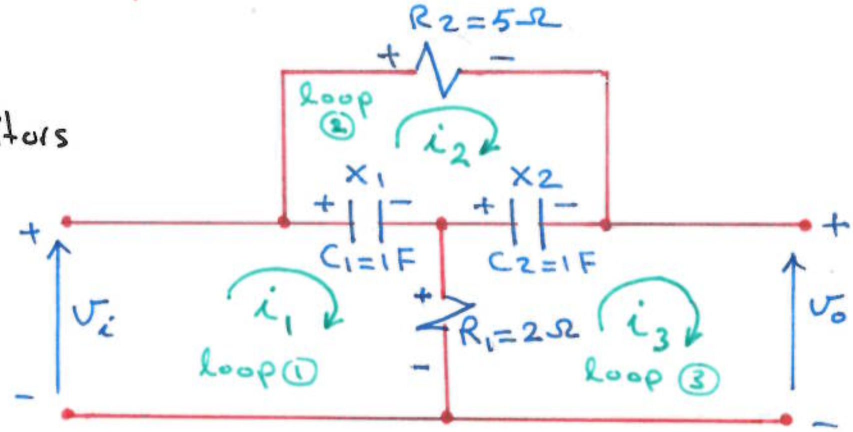
$$v_{o1} = y_1 = iR$$

and $v_{o2} = y_2 = v_c$

→ output equation will be

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}}_{C} \cdot \underbrace{\begin{bmatrix} i \\ v_c \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} \begin{bmatrix} u \end{bmatrix}$$

example: Find the state space and output equations for the Circuit shown below,



sol

we have two Capacitors

→ we have second order system,

we have:

$$v_{c_1} = x_1 \rightarrow \dot{v}_{c_1} = \dot{x}_1 \quad ; \text{and} \quad v_{c_2} = x_2 \rightarrow \dot{v}_{c_2} = \dot{x}_2$$

and $$i_{c_1} = c_1 \frac{dv_{c_1}}{dt} = c_1 \dot{v}_{c_1} = \dot{v}_{c_1} = \dot{x}_1$$

$$i_{c_2} = c_2 \frac{dv_{c_2}}{dt} = c_2 \dot{v}_{c_2} = \dot{v}_{c_2} = \dot{x}_2$$

now:

from loop ①

$$v_i = v_{c_1} + v_{R_1} = x_1 + (i_1 - i_3) R_1$$

note that $i_1 = i_{c_1} = \dot{x}_1 \; ; \; i_3 = i_{c_2} = \dot{x}_2$

$$\rightarrow v_i = x_1 + 2(\dot{x}_1 - \dot{x}_2) \text{ ----- } ①$$

from loop ②

$$v_{R_2} = v_{c_2} + v_{c_1}$$

$$i_{R_2} = i_{c_2} \rightarrow v_{R_2} = i_{c_2} R_2 = 5\dot{x}_2$$

$$\rightarrow 5\dot{x}_2 = x_2 + x_1 \rightarrow \dot{x}_2 = \frac{1}{5}(x_2 + x_1) \text{ ---- } ②$$

from loop ③

$$v_o = v_{R_1} - v_{c_2} \qquad ; v_{R_1} = (i_1 - i_3) R_1$$

$$\Rightarrow v_o = 2(\dot{x}_1 - \dot{x}_2) - x_2 \text{ ---- } ③$$

note that $v_o = v_i - v_{c_1} - v_{c_2}$

$$\rightarrow v_o = v_i - x_1 - x_2 \text{ ---- } ④$$

now sub, (2) in (1)

$$\rightarrow v_i = x_1 + 2(\dot{x}_1 - \dot{x}_2)$$

$$= x_1 + 2\left(\dot{x}_1 - \frac{1}{5}(x_2 + x_1)\right)$$

$\rightarrow V_i = 2\dot{x}_1 + \frac{3}{5}x_1 - \frac{2}{5}x_2$

$\rightarrow \dot{x}_1 = -\frac{3}{10}x_1 + \frac{1}{5}x_2 + \frac{1}{2}V_i$

and we have from (2)

$$\dot{x}_2 = \frac{1}{5}x_1 + \frac{1}{5}x_2 + 0\,V_i$$

then the state equation will be :

$$\underbrace{\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}}_{\dot{X}} = \underbrace{\begin{bmatrix}-\frac{3}{10} & \frac{1}{5}\\ \frac{1}{5} & \frac{1}{5}\end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix}x_1\\ x_2\end{bmatrix}}_{X} + \underbrace{\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}}_{B} \underbrace{\begin{bmatrix}V_i\end{bmatrix}}_{u}$$

and for output,

we have from (4)

$y = V_o = V_i - x_1 - x_2$

or $V_o = -x_1 - x_2 + V_i$

$\rightarrow$ in matrix form :

$$y = \underbrace{\begin{bmatrix}-1 & -1\end{bmatrix}}_{C} \cdot \underbrace{\begin{bmatrix}x_1\\ x_2\end{bmatrix}}_{X} + \underbrace{\begin{bmatrix}1\end{bmatrix}}_{D} \underbrace{\begin{bmatrix}V_i\end{bmatrix}}_{u}$$

Time Domain Analysis

The first important step in analyzing a control system is to derive a mathematical model of the system, then analysis will take place

In practice input signal to a control system is not known ahead of time but is random in nature, except in special cases.

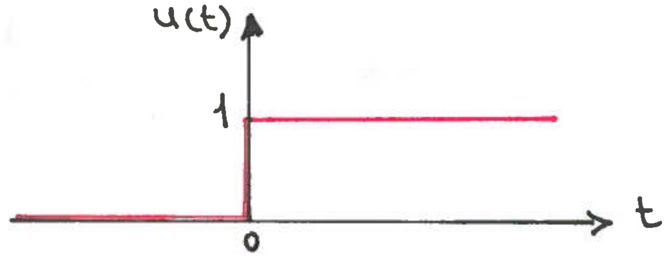
Many Design Criteria are based on the response to test signals or on the response to changes in initial contitions (without test signals).

Test Signals

1- Step function:

$u(t) = 1 \quad t \geq 0$

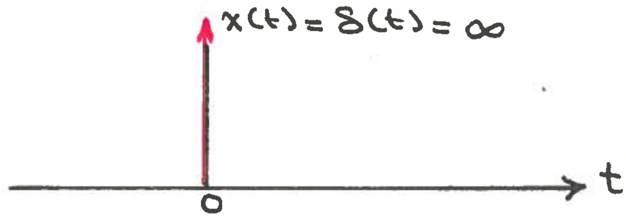
$u(t) = 0 \quad t < 0$



2- impulse function:

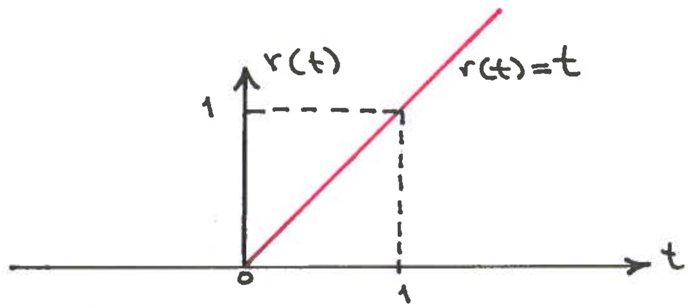
$x(t) = \delta(t) = \infty$ at $t=0$

$x(t) = 0$ elsewhere



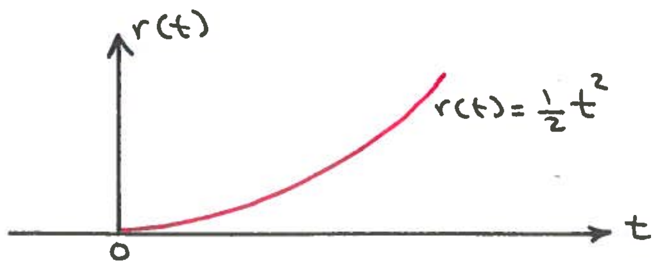
3- Ramp function:

$r(t) = t \quad t \geq 0$



4- Acceleration function (parabolic):

$$r(t) = \frac{1}{2} t^2$$



and other test signal like Sinusoidal functions and white noise.

System Time Response

Two Parts:

1- Transient response: response goes from initial state to final state.

2- steady state response: The manner in which the system output behaves as (t) approaches infinity.

$$\rightarrow c(t) = C_{tr}(t) + C_{ss}(t)$$

We say that a Control system is in equilibrium if in the absense of any disturbance or input, the output stays in the same state.

All control systems exhibit transients before steady state is attained because of energy storing elements.

A linear time invariant control system is stable if the output eventually comes back to its equilibrium state when the system subjected to an initial condition, and is critically stable if oscillations of the output continue forever.

Absolute stability : is whether the system is stable or unstable.

The system is unstable if the output diverges without bound from equilibrium state when the system is subjected to an initial condition.

Steady state Error

The steady state error of a system response is defined as the discrepancy between the output and the reference input when the steady state ($t \to \infty$) is reached.

Any physical control system inherently suffers steady state error in response to certain types of inputs. A system may have no steady state error to a step function input, but the same system may exhibit nonzero steady state error to a ramp input.

Impulse Response function:

for Linear time invariant system,

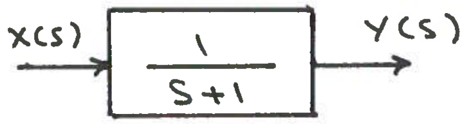
$$G(s) = \frac{Y(s)}{X(s)} \longrightarrow Y(s) = G(s)\, X(s)$$

since $X(s) = \mathcal{L}\, x(t) = \mathcal{L}\, \delta(t) = 1$

$\longrightarrow Y(s) = G(s) \longrightarrow y(t) = g(t)$, impulse response function.

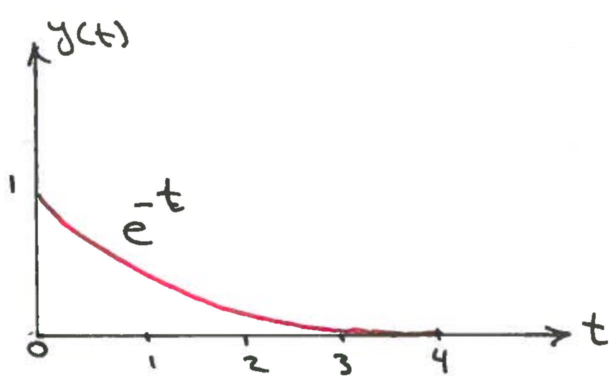
example: Given $G(s) = \frac{1}{s+1}$; find $y(t)$ for impulse input.

sol. $x(t) = \delta(t) \longrightarrow X(s) = 1$



$$Y(s) = 1 \cdot G(s) = \frac{1}{S+1}$$

$$\rightarrow y(t) = \mathcal{L}^{-1}\left(\frac{1}{S+1}\right) = e^{-t}$$



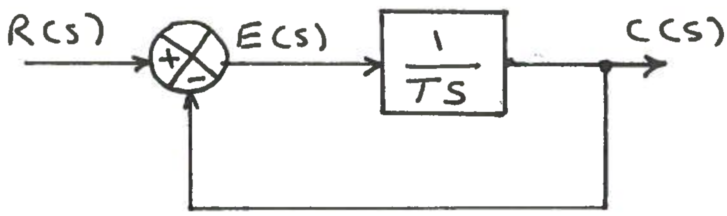
Unit Step Response :

$u(t) = 1$ for $t \geq 0$, and $u(t) = 0$ for $t < 0$

$$\rightarrow \mathcal{L}\, u(t) = \frac{1}{S} = R(s)$$

now Consider a unity F.B system with $G(s) = \frac{1}{TS}$

$$T.F = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)\cdot 1}$$



$$= \frac{1/TS}{1 + 1/TS} = \frac{1}{TS+1} = \frac{C(s)}{R(s)}$$

$R(s) = \frac{1}{S}$ for unit step input

$$\rightarrow C(s) = \frac{1}{TS+1} \cdot R(s) = \frac{1}{TS+1} \cdot \frac{1}{S}$$

and by using partial fraction expansion :

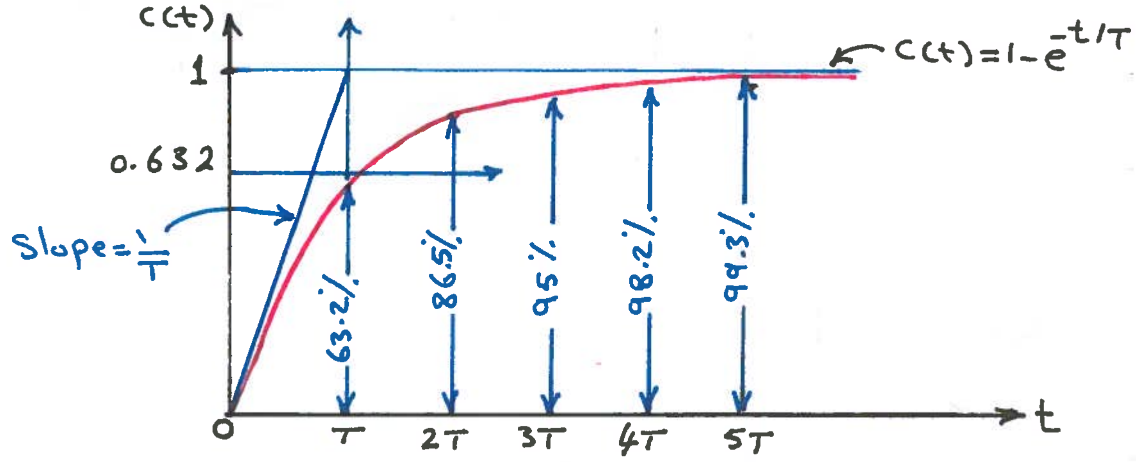
$$C(s) = \frac{1}{TS+1} \cdot \frac{1}{S} = \frac{A}{TS+1} + \frac{B}{S}$$

$$A = \lim_{S \to -\frac{1}{T}} \frac{1}{\cancel{(TS+1)}S} \cancel{(TS+1)} = -T$$

$$B = \lim_{S \to 0} \frac{1}{(TS+1)\cancel{S}} \cdot \cancel{S} = 1$$

$$\rightarrow c(t) = \mathcal{L}^{-1} C(s) = \mathcal{L}^{-1}\left(\frac{1}{S} - \frac{1}{S + \frac{1}{T}}\right) = 1 - e^{-t/T} \quad \text{for } t \geq 0$$

and T is the time constant of the system .



note that the smaller the time constant T, the faster the system response.

The slope of the tangent line at $t=0$ is $\frac{1}{T}$ since:

$$\frac{d\,c(t)}{dt} = \frac{1}{T} e^{-t/T} \Big|_{t=0} = \frac{1}{T}$$

and the slope of the response curve decreases from $\frac{1}{T}$ at $t=0$ to zero at $t=\infty$.

Unit Ramp Response:

$$r(t) = t \quad \text{for } t \geq 0$$

$$\rightarrow R(s) = \mathcal{L}\, r(t) = \frac{1}{s^2}$$

$$\rightarrow C(s) = \frac{1}{1+Ts} \cdot R(s) = \frac{1}{1+Ts} \cdot \frac{1}{s^2}$$

using partial fraction analysis

$$C(s) = \frac{A}{1+Ts} + \frac{B}{s} + \frac{C}{s^2}$$

$$A = \lim_{s \to -\frac{1}{T}} \frac{1}{(1\cancel{+Ts})} \cdot \frac{1}{s^2} (1\cancel{+}Ts) = T^2$$

$$C = \lim_{s \to 0} \frac{1}{(1+Ts)} \cdot \frac{1}{\cancel{s^2}} \cdot \cancel{s^2} = 1$$

for B:

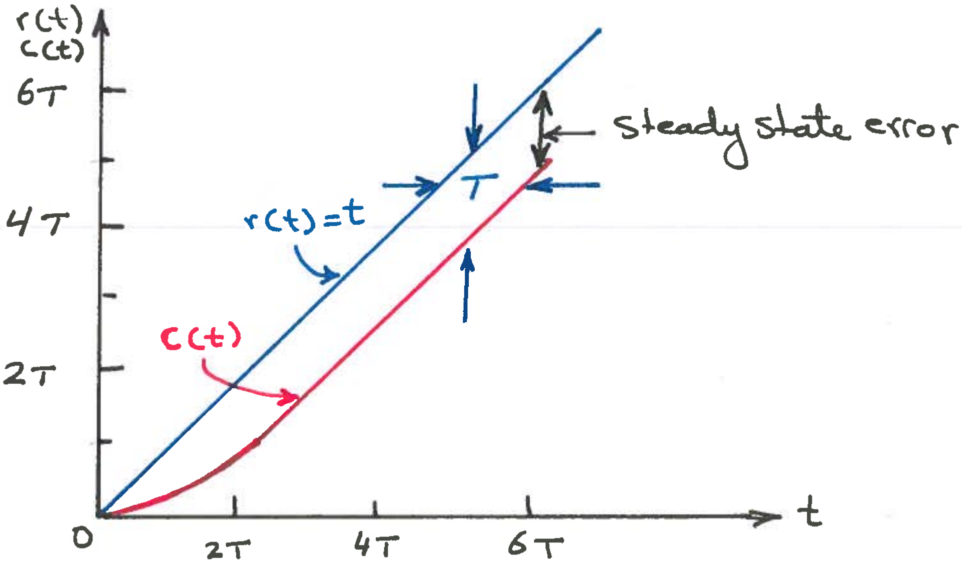
$$B = \lim_{s \to 0} \frac{1}{(1+Ts)} \cdot \frac{1}{s^{\cancel{2}}} \cancel{\cdot s} = \infty \rightarrow \text{not valid}$$

$$\rightarrow \frac{\partial}{\partial S}\left(\frac{1}{1+TS}\right) = \frac{0-T}{(1+TS)^2} = \frac{-T}{(1+TS)^2}$$

$$\Rightarrow B = \lim_{S \to 0} \frac{-T}{(1+TS)^2} = -T$$

$$\rightarrow C(S) = \frac{1}{S^2} - \frac{T}{S} + \frac{T^2}{1+TS} = \frac{1}{S^2} - \frac{T}{S} + T\frac{1}{S+1/T}$$

$$\rightarrow c(t) = \mathcal{L}^{-1} C(S) = t - T + T e^{-t/T}$$



The error Signal is :

$$e(t) = r(t) - c(t) = T(1 - e^{-t/T})$$

$\Rightarrow$ as t approaches ∞ , $e^{-t/T}$ approaches Zero,

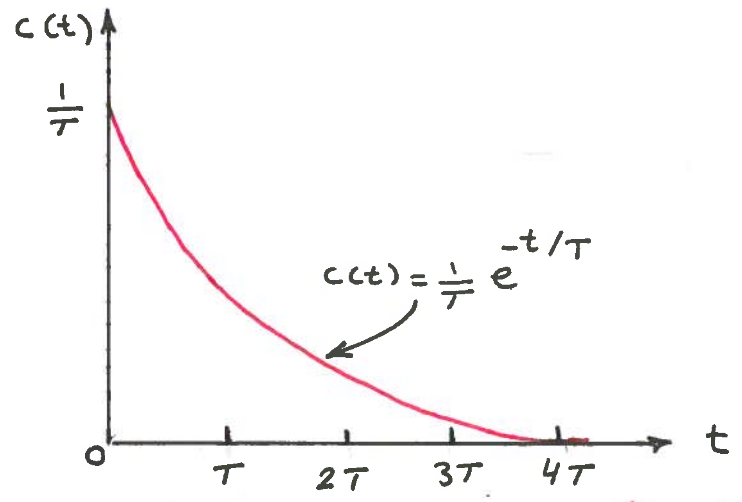
$\rightarrow$ $e(t)$ approaches T , or $e(\infty) = T$

Smaller Time Constant $\rightarrow$ Smaller Steady state error

Unit impulse Response :

$$r(t) = \delta(t) \rightarrow R(S) = 1$$

$$C(S) = \frac{1}{TS+1} \cdot 1 \rightarrow c(t) = \mathcal{L}^{-1} \frac{1}{T} \frac{1}{(S+\frac{1}{T})} = \frac{1}{T} e^{-t/T}$$



Important property of Linear Time Invariant System:

for Ramp input $\quad c(t) = t - T + T e^{-t/T} \qquad t \geq 0$

then: $\left.\frac{\partial}{\partial t} c(t)\right|_{ramp} = 1 - 0 + T e^{-t/T} \cdot \frac{-1}{T} = \underline{1 - e^{-t/T}}$ = unit step response.

also: $\left.\frac{\partial}{\partial t} c(t)\right|_{unit\ step} = 0 - e^{-t/T} \cdot \frac{-1}{T} = \underline{\frac{1}{T} e^{-t/T}}$ = impulse response

Second Order System:

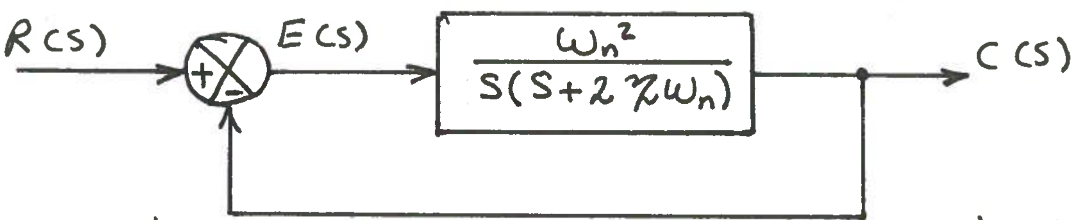
The standard form of the second Order system is :

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

where:

ζ : is the damping ratio of the actual damping to the critical damping ,

ω_n : is the undamped natural frequency



The System above can be Solved to a unit step in three cases as following:

1- Underdamped case where $0 < \zeta < 1$

in this case $\frac{C(S)}{R(S)} = \frac{\omega_n^2}{(S+\zeta\omega_n+j\omega_d)(S+\zeta\omega_n-j\omega_d)}$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and called the damped natural frequency.

for a unit step input, $c(t)$ can be written as:

$$c(t) = 1 - e^{-\zeta\omega_n t}\left\{\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right\}$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left\{\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right\} \quad \text{for } t \geq 0$$

2- Critically damped case where $\zeta = 1$

The response to step input can be written as:

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t) \quad \text{for } t \geq 0$$

3- Overdamped case where $\zeta > 1$

The response to step input is:

$$c(t) = 1 - e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t} \quad \text{for } t \geq 0$$

Second Order System and Transient Response Specifications:

Rise time t_r: The time required for the response to rise from 10% to 90%, or 0% to 100% of its final value.

For underdamped second order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.

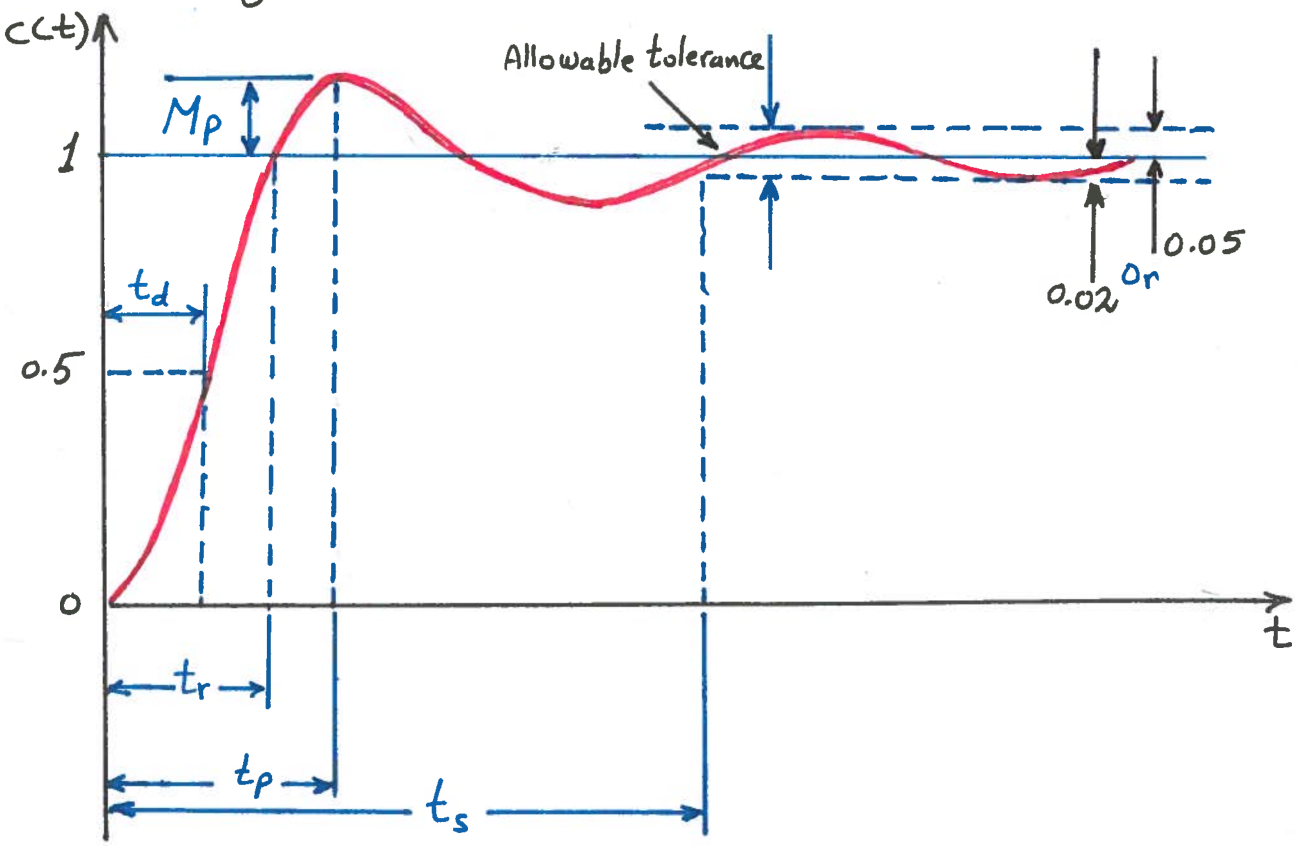
Delay Time t_d : The time required for the step response to reach 50% of its final value.

Peak Time t_p: The time required for the response to reach the first peak of the overshoot.

Maximum (percent) overshoot, M_p : is the maximum peak value of the response curve measured from unity. If the final steady state value of the response differs from unity, then it is common to use the maximum percent overshoot:

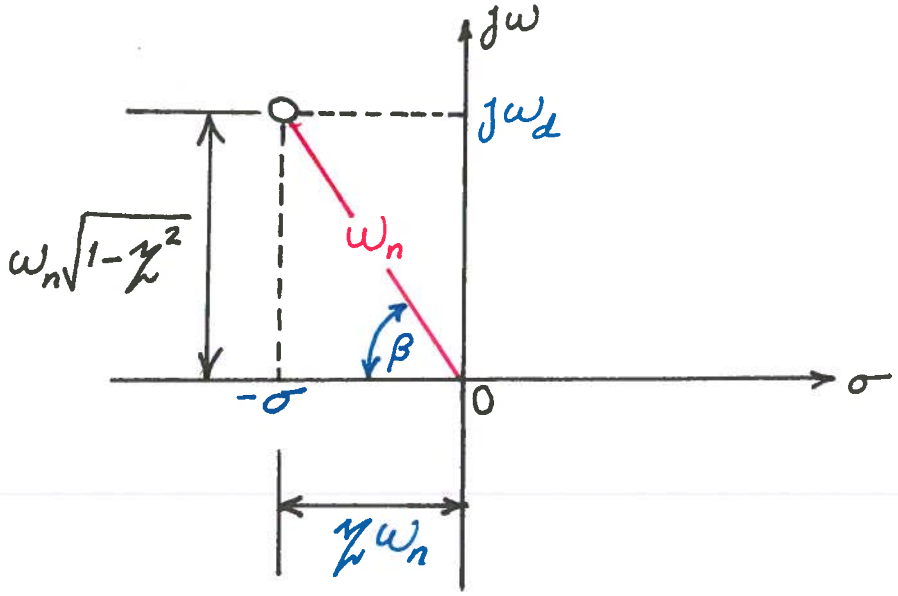
$$\text{Percent Maximum Overshoot} = \frac{c(t)_{max} - c(t)_{ss}}{c(t)_{ss}}$$

Settling Time t_s : is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value usually 2% or 5% .



now, for step input, and final value is unity (1)

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma}\right)$$



Peak time t_p ; $t_p = \dfrac{\pi}{\omega_d}$

$$\longrightarrow M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = e^{-\pi\sigma/\omega_d}$$

and settling time t_s,

$$t_s = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \quad (\text{for } 2\% \text{ Criterion})$$

or $$t_s = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n} \quad (\text{for } 5\% \text{ Criterion})$$

example: Consider a system with $\zeta = 0.6$ and $\omega_n = 5$ rad/sec. obtain the rise time (t_r), peak time (t_p), maximum overshoot (M_p), and settling time (t_s) when the system is subjected to a unit step input.

Sol: $\zeta = 0.6$, $\omega_n = 5$

$$\rightarrow \omega_d = \omega_n\sqrt{1-\zeta^2} = 5 * \sqrt{1-(0.6)^2} = 4$$

$$\sigma = \zeta\omega_n = 0.6 \times 5 = 3$$

$$\therefore \tan\beta = \frac{\omega_d}{\sigma} \Rightarrow \beta = \tan^{-1}\frac{\omega_d}{\sigma} = 0.93 \text{ rad}$$

$$\rightarrow t_r = \frac{\pi-\beta}{\omega_d} = \frac{3.14-0.93}{4} = 0.55 \text{ Sec}$$

$$\rightarrow t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785 \text{ sec}$$

$$\rightarrow M_p = e^{-\pi\sigma/\omega_d} = e^{-3.14(3/4)} = 0.095$$

$\rightarrow$ maximum percentage overshoot is 9.5%

$$\rightarrow t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33 \text{ sec} \quad \text{for } 2\% \text{ Criterion}$$

$$\rightarrow t_s = \frac{3}{\sigma} = \frac{3}{3} = 1 \text{ sec} \quad \text{for } 5\% \text{ Criterion.}$$

Impulse Response of Second Order Systems:

For a unit impulse input $r(t)$, $R(s)=1$

$$C(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

For $0 \leq \zeta < 1$

$$c(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\omega_n\sqrt{1-\zeta^2}\, t \quad , \text{for } t \geq 0$$

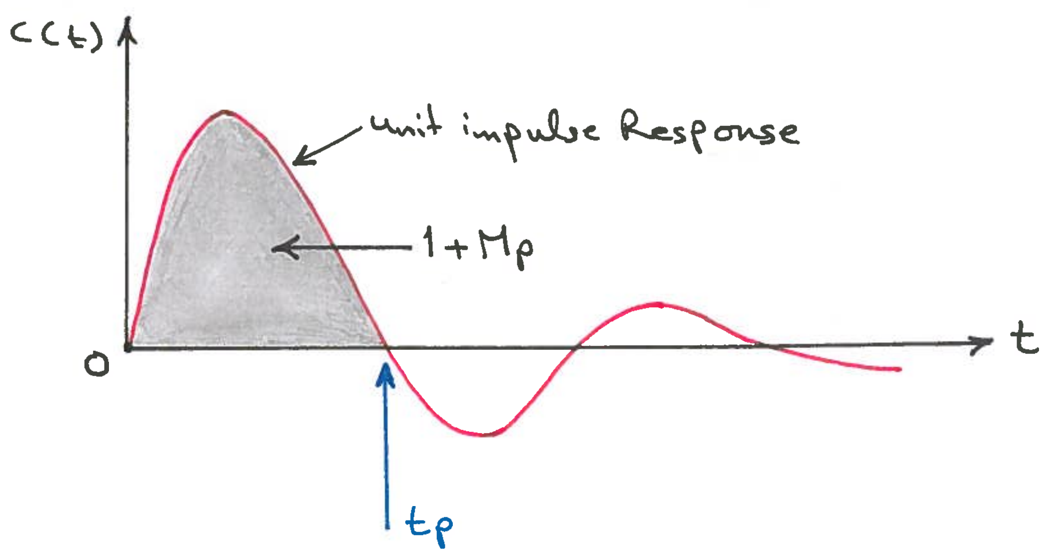
For $\zeta = 1$, $c(t) = \omega_n^2 t e^{-\omega_n t}$, for $t \geq 0$

For $\zeta > 1$

$$c(t) = \frac{\omega_n}{2\sqrt{\zeta^2-1}} e^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t} - \frac{\omega_n}{2\sqrt{\zeta^2-1}} e^{-(\zeta+\sqrt{\zeta^2-1})\omega_n t},$$

for $t \geq 0$

note that we can also obtain the time response $c(t)$ by differentiating the corresponding unit step response, since the unit impulse function is the time derivative of the unit step function.



The area under the unit impulse curve from t=0 to the time of the first Zero is $1+M_p$, where M_p is the maximum Overshoot for the unit step response.

The peak time t_p (for the unit step response) corresponds to the time that the unit impulse response first Crosses the time axis.

Maximum overshoot for the unit impulse response of the under damped system occurs at:

$$t = \frac{\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n\sqrt{1-\zeta^2}}, \quad 0<\zeta<1$$

and Maximum overshoot is:

$$c(t)_{max} = \omega_n \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right), \quad 0<\zeta<1$$

again, for unit step

1- underdamped case $\quad 0<\zeta<1$

$$c(t)=\mathcal{L}^{-1}C(s)=1-e^{-\zeta\omega_n t}\left(\cos\omega_d t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) \quad , t\geq 0$$

where $\omega_d=\omega_n\sqrt{1-\zeta^2}$

2- Critically damped case $\quad \zeta=1$

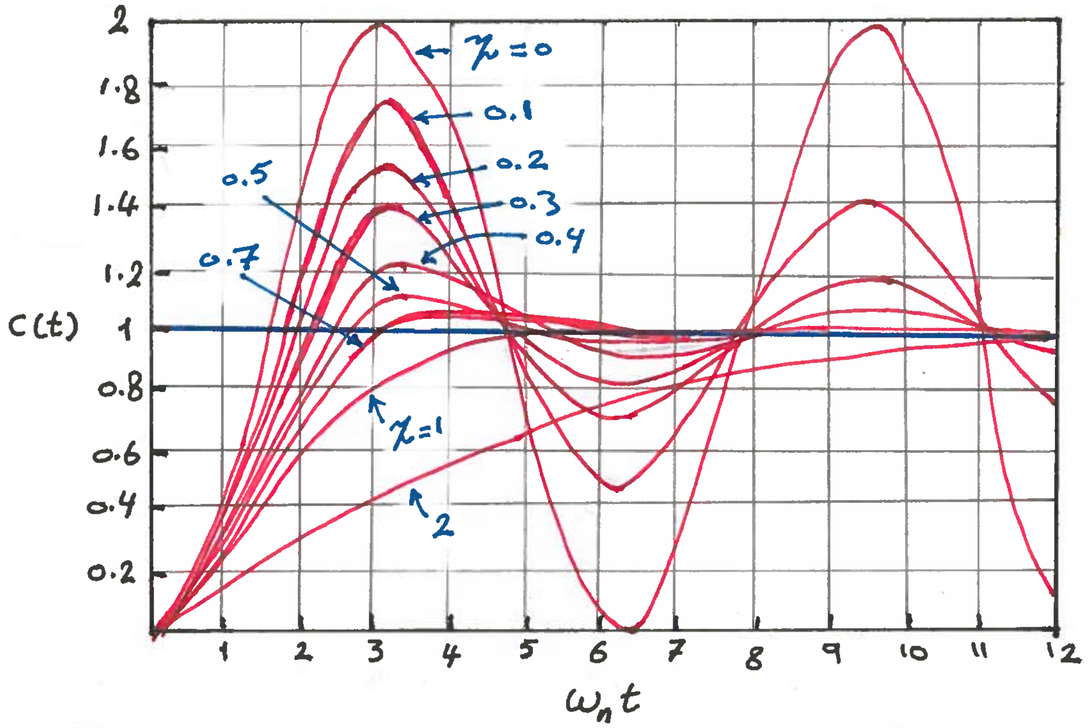
$$c(t)=\mathcal{L}^{-1}C(s)=1-e^{-\omega_n t}(1+\omega_n t) \quad , \quad t\geq 0$$

3- over damped case $\quad \zeta>1$

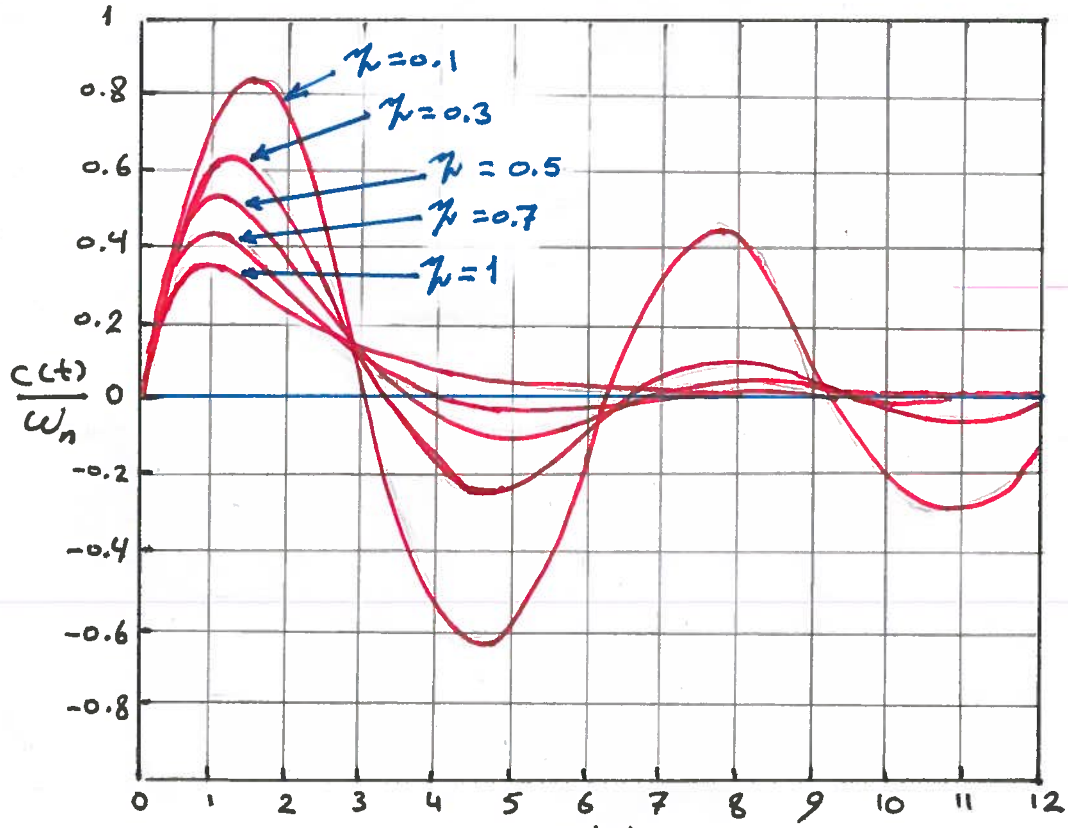
$$c(t)=\mathcal{L}^{-1}C(s)=1-e^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t} \quad , t\geq 0$$

4- Zero damping case $\quad \zeta=0$

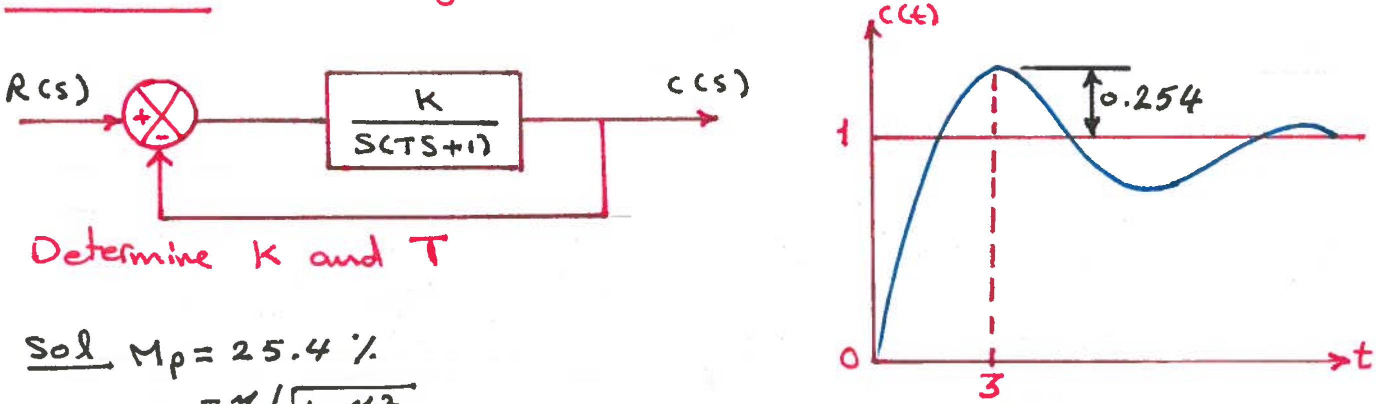
$$c(t)=\mathcal{L}^{-1}C(s)=1-\cos\omega_n t \quad , \quad t\geq 0$$



and for impulse response,



example: Given a system and its output response as below:



Determine K and T

Sol. $M_p = 25.4\ \%$

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 0.254 \rightarrow \ln(0.254) = \frac{-\pi\zeta}{\sqrt{1-\zeta^2}} \rightarrow \zeta = 0.4$$

$$t_p \text{ from Curve } = 3 \text{ sec} = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \rightarrow \omega_n = 1.14 \text{ rad/sec}$$

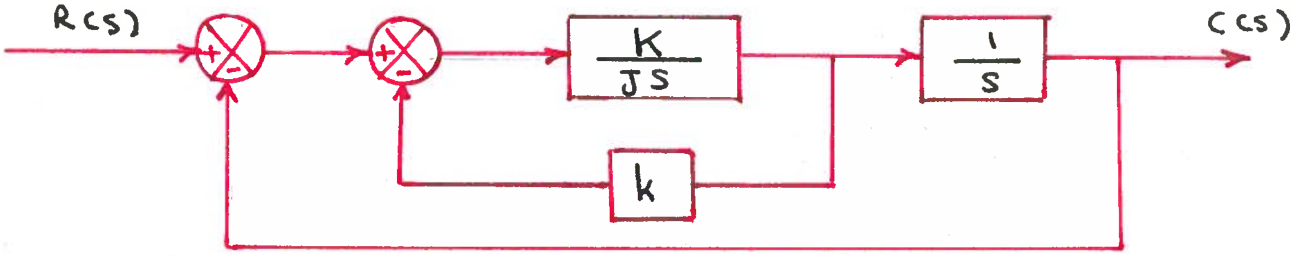
$$\frac{C(s)}{R(s)} = \frac{K}{TS^2+S+K}$$

$$\rightarrow \omega_n = \sqrt{\frac{K}{T}} \quad , \quad 2\zeta\omega_n = \frac{1}{T}$$

$$\rightarrow T = \frac{1}{2\zeta\omega_n} = 1.09$$

$$\text{and } K = \omega_n^2\, T = 1.42$$

example: Determine the values of K and k of the closed loop system shown, so that $M_p = 25\%$ and peak time is 2 sec. assume that $J = 1$.



sol:

$$\frac{C(s)}{R(s)} = \frac{K}{JS^2 + Kk S + K} \quad , J = 1$$

$$\rightarrow \omega_n = \sqrt{K} \quad , \quad 2\zeta\omega_n = Kk$$

$$M_p = e^{-\zeta\pi / \sqrt{1-\zeta^2}} = 0.25 \longrightarrow \zeta = 0.404$$

$$t_p = 2 = \frac{\pi}{\omega_d} \Rightarrow \omega_d = 1.57 \text{ rad/sec}$$

$$\longrightarrow \omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 1.72 \text{ rad/sec}$$

$$K = \omega_n^2 = 2.95 \quad , \text{ and } k = \frac{2\zeta\omega_n}{K} = 0.471$$

example: Consider the closed loop system:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

find the values of ζ and ω_n so that the system responds to a step input with 5% overshoot and settles after 2 sec (use the 2% criterion).

sol:

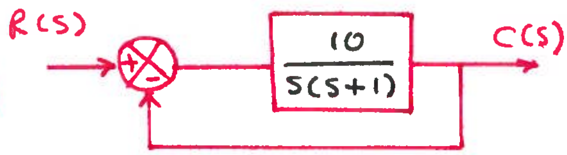
$$M_p = e^{-\pi\zeta / \sqrt{1-\zeta^2}} = 0.05 \rightarrow \frac{-\pi\zeta}{\sqrt{1-\zeta^2}} = \ln(0.05)$$

$$\longrightarrow \zeta = 0.69$$

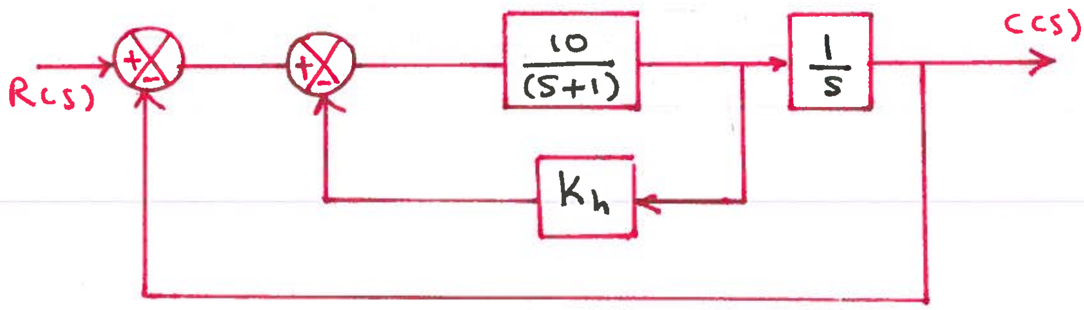
$$t_s = 2 = \frac{4}{\zeta\omega_n} \longrightarrow \omega_n = 2.898 \text{ rad/sec}$$

example: Given the system below:

The damping ratio of the system is 0.158 and the undamped natural frequency is 3.16 rad/sec.



The system was improved by employing a feed back system as shown:

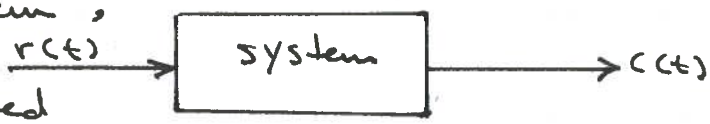


Determine the value of K_h so that the damping ratio of the system is 0.5.

Ans: $K_h = 0.216$

Steady state Error:

Assume the below system,

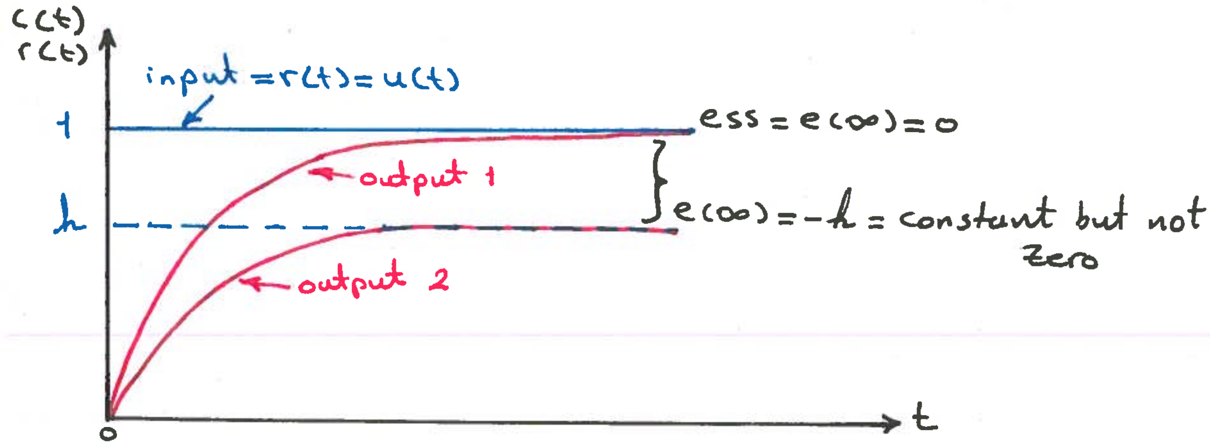


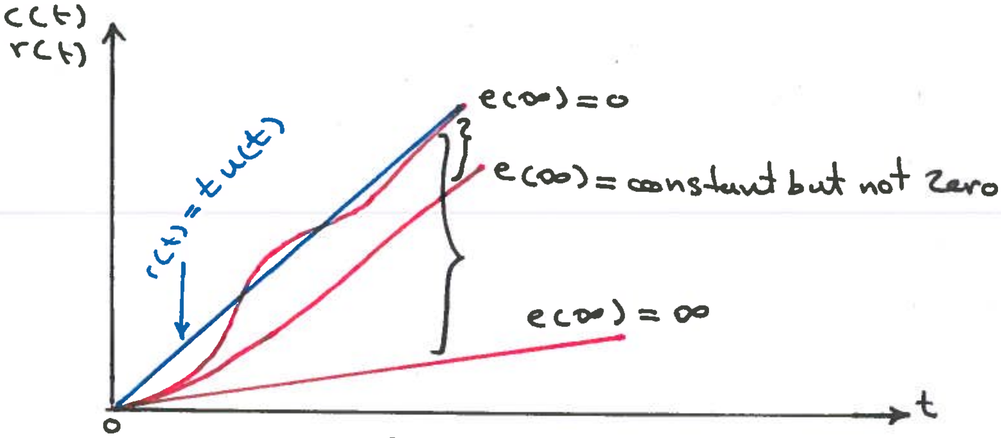
the input r(t) is the desired behavior of the system, (reference), and the output c(t) is the actual behavior.

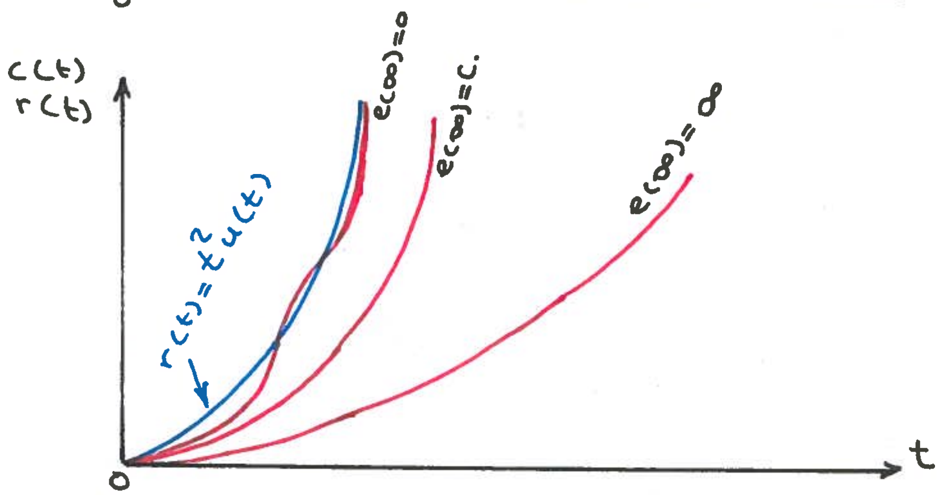
then error signal $e(t) = c(t) - r(t)$

and then $e(\infty)$: steady state Error

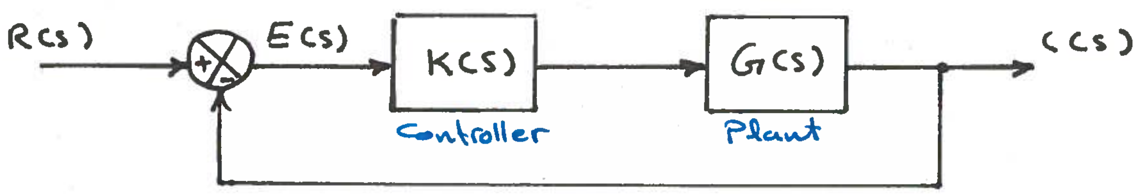
now for different test input signals, the output could be as following:





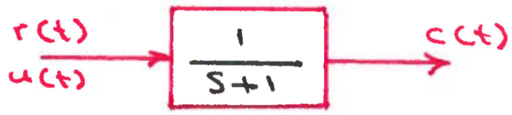


For unity Feedback system :



when $c(t)$ is found, $e(t)$ can be calculated and if its limit as $t \to \infty$ is calculated we can find $e(\infty)$.

example: given the first order System below, our aim is to find the Steady state error.



Sol: $R(s) = \frac{1}{s}$

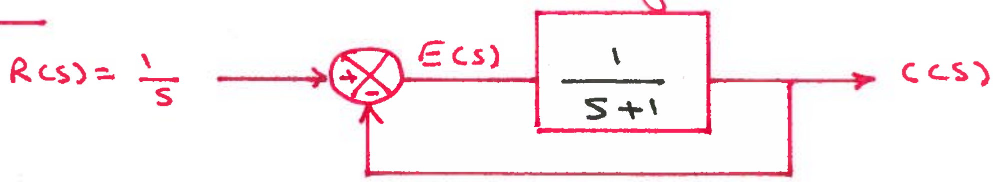
$$c(s) = \frac{1}{(s+1)} \cdot \frac{1}{s}$$; using Partial fraction Expansion,

$$\rightarrow c(t) = (1 - e^{-t})u(t)$$

$$\rightarrow e(t) = c(t) - r(t) = -e^{-t} u(t)$$

$$\Rightarrow \lim_{t \to \infty} e(t) = 0 \rightarrow e(\infty) = e_{ss} = 0$$

example: find the S.S.E for the system shown,



note: Loop Gain fuction (L(s)) is the product of all of the transfer functions around the loop.

sol. $\frac{C(s)}{R(s)} = \frac{1}{s+2}$; $R(s) = \frac{1}{s}$

$$\rightarrow c(t) = (0.5 - 0.5e^{-2t})u(t)$$

$$\rightarrow \lim_{t \to \infty} e(t) = -0.5 \Rightarrow e(\infty) \neq$$ but constant, (stable)

* In General, for any system with any input, Final value theorem states that:

$$\text{If } \mathcal{L}\{e(t)\} = E(s)$$

$$\text{then } \mathcal{L}\{e(\infty)\} = \lim_{s \to 0} s E(s)$$

note: This theorem holds only if signal E(s) has poles only in the left half plane or at the origin. Otherwise the error either blows up due to instability or keeps oscillating without settling.

For a general system having $L(s) = G(s)H(s)$,

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)} = \frac{1}{1+L(s)}$$

$$E_{ss} = \mathcal{L}\{e(\infty)\} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \cdot \frac{1}{1+L(s)} \cdot R(s) \ldots$$

... eq. (1)

now, if $R(s) = \frac{1}{s}$ (step input),

$$E_{ss} = \lim_{s\to 0} \cancel{s} \frac{1}{1+L(s)} \cdot \frac{1}{\cancel{s}} = \lim_{s\to 0} \frac{1}{1+L(s)}$$

$$= \frac{1}{1+\lim_{s\to 0} L(s)}$$

for $e(\infty) = 0$, this part from last equation ($\lim_{s\to 0} L(s)$) must go to ∞.

$\rightarrow e(\infty) = 0$ if $\lim_{s\to 0} L(s) = \infty$, $n \geq 1$ (only)

Therefore if $L(s) = \dfrac{(s+z_1)(s+z_2)\cdots}{s^n (s+P_1)(s+P_2)\cdots}$, $n \geq 0$ in General

Then, For a unit step input, has $e(\infty) = 0$ only if the loop gain function contains at least one pure integrator, (pole at the origin), i.e. if $n \geq 1$ where n is the order of the pole at origin.

For Ramp Input , $R(s) = \frac{1}{s^2}$

$$\rightarrow E_{ss} = \lim_{s\to 0} s \frac{1}{1+L(s)} \cdot \frac{1}{s^2} = \lim_{s\to 0} \frac{1}{s+sL(s)}$$

$$= \lim_{s\to 0} \frac{1}{sL(s)}$$

So in order to have $e(\infty) = 0$ for ramp input, the $L(s)$ has to have at least two pure integrators, i.e., $n \geq 2$.

For parabolic input, $R(s) = \frac{1}{s^3}$,

$$\rightarrow E_{ss} = \lim_{s \to 0} s \frac{1}{1+L(s)} \cdot \frac{1}{s^3} = \lim_{s \to 0} \frac{1}{s^2 + s^2 L(s)}$$

$$= \lim_{s \to 0} \frac{1}{s^2 L(s)}$$

then, In order to have $e(\infty) = 0$, need the $L(s)$ having at least three integrators, i.e. $n \geq 3$

Static Error Constants:

Position Constant: $K_p = \lim_{s \to 0} L(s)$; $e_{step}(\infty) = \frac{1}{1+K_p}$

velocity constant: $K_v = \lim_{s \to 0} s L(s)$; $e_{ramp}(\infty) = \frac{1}{K_v}$

Acceleration Constant: $K_a = \lim_{s \to 0} s^2 L(s)$; $e_{Parabola}(\infty) = \frac{1}{K_a}$

and these constants either 0 or constant or ∞

System type:

System type is defined as the number of pure integrators in the $L(s)$, i.e.:

$$L(s) = \frac{(s+z_1)(s+z_2)\cdots}{s^n (s+p_1)(s+p_2)\cdots}$$

→ The number of repeated poles at the origin is the system type.

If we write the polynomial of $L(s)$ as below

$$L(s) = \frac{K(a_q s^q + a_{q-1} s^{q-1} + \cdots a_1 s + 1)}{s^n (b_\ell s^\ell + b_{\ell-1} s^{\ell-1} + \cdots b_1 s + 1)}$$

→ n is the type of the system,

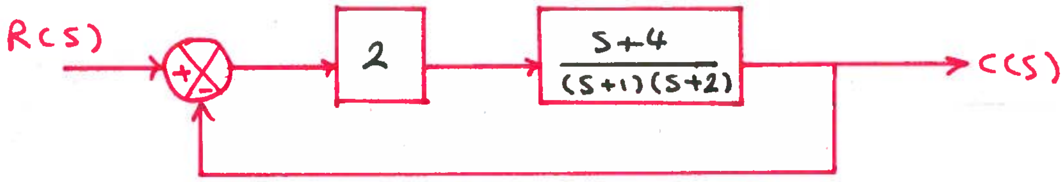
K is the static error constant.

if $n=0 \longrightarrow K=K_p$

$n=1 \longrightarrow K=K_v$

$n=2 \longrightarrow K=K_a$

example: Find the system type and static error constants for the below system,



<u>sol</u>

$$L(s) = \frac{2(s+4)}{(s+1)(s+2)} = \frac{8(\frac{1}{4}s+1)}{2(\frac{1}{2}s^2+\frac{3}{2}s+1)}$$

$\rightarrow K=\frac{8}{2}=4$

and system is type 0

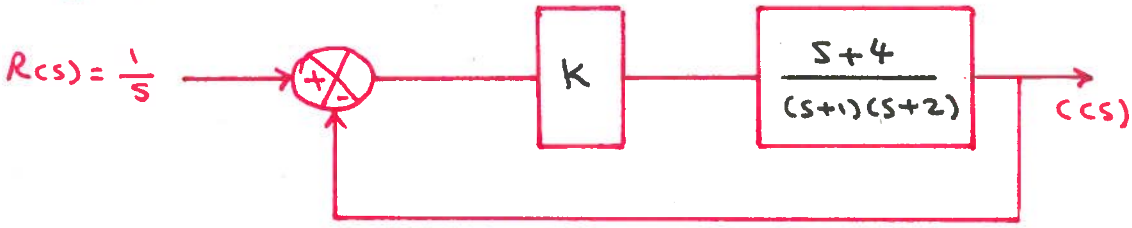
$\rightarrow K=4=K_p$

$$K_v = \lim_{s\to 0} s \frac{2(s+4)}{(s+1)(s+2)} = 0$$

$$K_a = \lim_{s\to 0} s^2 \frac{2(s+4)}{(s+1)(s+2)}$$

System type	Step Input	Ramp Input	Parabola Input
$n=0$	K_p = constant $e(\infty)=\frac{1}{1+K_p}$	$K_v=0$ $e(\infty)=\infty$	$K_a=0$ $e(\infty)=\infty$
$n=1$	$K_p=\infty$ $e(\infty)=0$	K_v = constant $e(\infty)=\frac{1}{K_v}$	$K_a=0$ $e(\infty)=\infty$
$n=2$	$K_p=\infty$ $e(\infty)=0$	$K_v=\infty$ $e(\infty)=0$	K_a = constant $e(\infty)=\frac{1}{K_a}$

example: Design a controller K such that $e_{ss} \leq 0.2$ for a unit step input,



sol :

$$L(s)=\frac{K(s+4)}{(s+1)(s+2)}=\frac{K(s+4)}{s^2+3s+2}=\frac{4K}{2}\,\frac{\frac{1}{4}s+1}{\frac{1}{2}s^2+\frac{3}{2}s+1}$$

$\rightarrow n=0$

$$K_p=\frac{4K}{2}=2K \;;\quad e(\infty)=\frac{1}{1+K_p}=\frac{1}{1+2K}$$

$\because e(\infty)\leq 0.2 \rightarrow K\geq 2$

Table A–1 Laplace Transform Pairs

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{s^n}$
5	$t^n \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!} t^{n-1}e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
11	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
12	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$
13	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$
14	$\frac{1}{a}\left(1 - e^{-at}\right)$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}\left(e^{-at} - e^{-bt}\right)$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}\left(be^{-bt} - ae^{-at}\right)$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}\left[1 + \frac{1}{a-b}\left(be^{-at} - ae^{-bt}\right)\right]$	$\frac{1}{s(s+a)(s+b)}$

(continues on next page)

Table A–1 (*continued*)

18	$\frac{1}{a^2}(1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
21	$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}\,t \quad (0<\zeta<1)$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
23	$-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t - \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0<\zeta<1,\quad 0<\phi<\pi/2)$	$\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}$
24	$1-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t + \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0<\zeta<1,\quad 0<\phi<\pi/2)$	$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$
25	$1 - \cos\omega t$	$\frac{\omega^2}{s(s^2+\omega^2)}$
26	$\omega t - \sin\omega t$	$\frac{\omega^3}{s^2(s^2+\omega^2)}$
27	$\sin\omega t - \omega t\cos\omega t$	$\frac{2\omega^3}{(s^2+\omega^2)^2}$
28	$\frac{1}{2\omega}t\sin\omega t$	$\frac{s}{(s^2+\omega^2)^2}$
29	$t\cos\omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
30	$\frac{1}{\omega_2^2-\omega_1^2}(\cos\omega_1 t - \cos\omega_2 t) \qquad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2+\omega_1^2)(s^2+\omega_2^2)}$
31	$\frac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t)$	$\frac{s^2}{(s^2+\omega^2)^2}$

Table A–2 Properties of Laplace Transforms

1	$\mathcal{L}[Af(t)] = AF(s)$
2	$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
3	$\mathcal{L}_\pm\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0\pm)$
4	$\mathcal{L}_\pm\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0\pm) - \dot{f}(0\pm)$
5	$\mathcal{L}_\pm\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}\overset{(k-1)}{f}(0\pm)$ where $\overset{(k-1)}{f}(t) = \frac{d^{k-1}}{dt^{k-1}}f(t)$
6	$\mathcal{L}_\pm\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t)\,dt\right]_{t=0\pm}$
7	$\mathcal{L}_\pm\left[\int\cdots\int f(t)(dt)^n\right] = \frac{F(s)}{s^n} + \sum_{k=1}^{n}\frac{1}{s^{n-k+1}}\left[\int\cdots\int f(t)(dt)^k\right]_{t=0\pm}$
8	$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$
9	$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s)$ if $\int_0^\infty f(t)\,dt$ exists
10	$\mathcal{L}[e^{-at}f(t)] = F(s+a)$
11	$\mathcal{L}[f(t-\alpha)1(t-\alpha)] = e^{-\alpha s}F(s) \qquad \alpha \geq 0$
12	$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$
13	$\mathcal{L}[t^2f(t)] = \frac{d^2}{ds^2}F(s)$
14	$\mathcal{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \qquad (n = 1, 2, 3, \dots)$
15	$\mathcal{L}\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s)\,ds$ if $\lim_{t\to 0}\frac{1}{t}f(t)$ exists
16	$\mathcal{L}\left[f\left(\frac{1}{a}\right)\right] = aF(as)$
17	$\mathcal{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau\right] = F_1(s)F_2(s)$
18	$\mathcal{L}[f(t)g(t)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)G(s-p)\,dp$

Finally, we present two frequently used theorems, together with Laplace transforms of the pulse function and impulse function.

Initial value theorem	$f(0+) = \lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s)$
Final value theorem	$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$
Pulse function $f(t) = \frac{A}{t_0}1(t) - \frac{A}{t_0}1(t - t_0)$	$\mathscr{L}[f(t)] = \frac{A}{t_0 s} - \frac{A}{t_0 s}e^{-st_0}$
Impulse function $g(t) = \lim_{t_0\to 0} \frac{A}{t_0}, \quad \text{for } 0 < t < t_0$ $= 0, \quad \text{for } t < 0, t_0 < t$	$\mathscr{L}[g(t)] = \lim_{t_0\to 0}\left[\frac{A}{t_0 s}(1 - e^{-st_0})\right]$ $= \lim_{t_0\to 0} \frac{\frac{d}{dt_0}[A(1 - e^{-st_0})]}{\frac{d}{dt_0}(t_0 s)}$ $= \frac{As}{s} = A$

example: Find all the time domain specifications ($\zeta, \omega_n, \omega_d, t_r, t_p, t_s, M_p$) for a unity feedback control system whose open loop transfer function is given by:

$$G(s) = \frac{25}{S(S+6)}$$

sol:
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{25}{S^2+6S+25}$$

$\omega_n^2 = 25 \longrightarrow \omega_n = 5$ rad/sec

$2\zeta\omega_n = 6 \longrightarrow \zeta = 0.6$

$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4$ rad/sec

$\beta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = 53.13^\circ = 0.92$ rad

$t_r = \frac{\pi-\beta}{\omega_d} = 0.55$ sec

$t_p = \frac{\pi}{\omega_d} = 0.785$ sec

$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 9.5\ \%$

$t_s = \frac{4}{0.6\times5} = 1.33$ sec for 2% criterion

example: Determine the position, velocity, and acceleration error constants for a unity feedback control system whose open loop transfer function is given by: $G(s) = \frac{K}{S(S+4)(S+10)}$.
For $K = 400$, determine the steady state error for a unit ramp input.

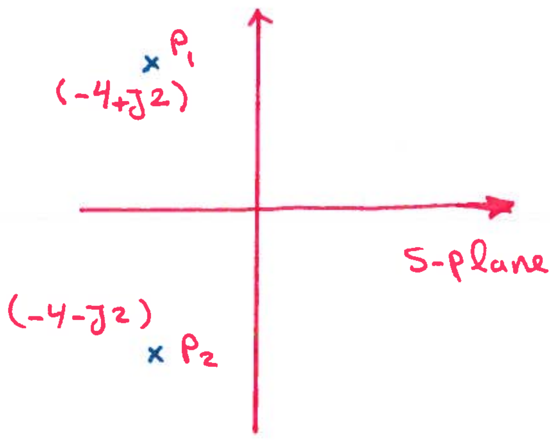
sol: $K_p = \lim_{s\to0} G(s) = \infty$; $K_v = \lim_{s\to0} sG(s) = \frac{K}{40}$

and $K_a = \lim_{s\to0} s^2 G(s) = 0$

For a unit ramp input, when $K = 400$

$$E_{ss} = \frac{1}{K_v} = \frac{1}{K/40} = \frac{40}{400} = \frac{1}{10}$$

example: Given the closed loop poles of a system as below: Find the unit step response of the system and the settling time for 2% tolerance.



Sol.

characteristic equation is

$$(s+4-j2)(s+4+j2) = s^2+8s+20$$

$$\rightarrow 2\zeta\omega_n = 8 \quad , \quad \omega_n = \sqrt{20} = 4.472 \text{ rad/sec}$$

$$\rightarrow \zeta = 0.9$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 2$$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$= 1 - \frac{e^{-4t}}{0.44} \sin\left(2t + \tan^{-1}\frac{0.44}{0.9}\right)$$

$$= 1 - 2.27\, e^{-4t} \sin(2t + 0.144\pi)$$

$$t_s = \frac{4}{\zeta\omega_n} = 1 \text{ sec} .$$

example: The open loop transfer function of a feedback control system is given by: $G(s)H(s) = \dfrac{K(s+1)}{s(1+Ts)(1+2s)}$

Determine the error coefficients and the errors due to the unit positional input, unit ramp input, and unit parabolic input if $K=10$ and $T=4$

Sol: $G(s)H(s) = \dfrac{10(s+1)}{s(1+4s)(1+2s)}$

1- $K_p = \lim_{s\to 0} G(s)H(s) = \infty$

2- $K_v = \lim_{s \to 0} s\,G(s)H(s) = 10$

3- $K_a = \lim_{s \to 0} s^2\,G(s)H(s) = 0$

now E_{ss}: for step input $= \frac{1}{1+K_p} = 0$

for unit ramp input $= \frac{1}{K_v} = 0.1$

for unit parabolic input $= \frac{1}{K_a} = \infty$

Routh's Stability Criterion:

Routh's stability (Routh-Hurwitz) Criterion is a way that can be applied to the characteristic equation to find out the presence of the roots without having to solve the actual equation.

The necessary conditions for a system to be stable are:

1- Non of the coefficients of the characteristic equation should be missing or Zero.

2- All the Coefficients should be real and should have the same sign.

The sufficient condition for a system to be stable is that each term of the first column of Routh's array be positive and should have the same sign.

Routh's array for the characteristic equation:

$$\frac{C(s)}{R(s)} = \frac{r_o S^m + r_1 S^{m-1} + \dots r_{m-1} S + r_m}{a_o S^n + a_1 S^{n-1} + \dots + a_{n+} S + a_n}$$

$\rightarrow$ c/cs. eq. $= a_o S^n + a_1 S^{n-1} + a_2 S^{n-2} + \dots + a_{n-1} S + a_n = 0$

For $n=7$, Routh's array can be formed as below:

$$\begin{vmatrix} a_0 & a_2 & a_4 & a_6 \\ a_1 & a_3 & a_5 & a_7 \\ b_1 & b_3 & b_5 & \\ c_1 & c_3 & & \\ d_1 & d_3 & & \\ e_1 & & & \\ f_1 & & & \end{vmatrix}$$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \qquad c_3 = \frac{b_1 a_5 - a_1 b_5}{}$$

$$b_3 = \frac{a_1 a_4 - a_0 a_5}{a_1} \qquad d_1 = \frac{c_1 b_3 - b_1 c_3}{c_1}$$

$$b_5 = \frac{a_1 a_6 - a_0 a_7}{a_1} \qquad d_3 = \frac{c_1 b_5 - b_1 \times 0}{c_1}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_3}{b_1} \qquad e_1 = \frac{d_1 c_3 - c_1 d_3}{d_1}$$

$$f_1 = \frac{e_1 d_3 - d_1 \times 0}{e_1}$$

Limitations:

1- Routh's stability criterion is valid only if the characteristic equation is algebric, and if any Coefficient in the equation is complex or contains power of "e", this Criterion cannot be applied.

2- It gives us an information as to how many roots are lying in the right hand side of the S-plane. Values of the roots are not available, also it cannot distinguish between real and complex roots.

If any of the coefficients is zero or negative in the presence of at least one positive coefficient, there is a root or roots that are imaginary or that have positive real parts. Therefore, in such a case the system is not stable.

example: Consider the following polynomial

$$S^4 + 2S^3 + 3S^2 + 4S + 5 = 0$$

* if any coefficient is missing, it may be replaced by zero in the array

S^4	1	3	5
S^3	2	4	0 → Can be devided by 2
	1	2	0
S^2	1	5	
S^1	-3		
S^0	5		

The number of changes in sign of the coefficients in the first Column is 2, this means that there are two roots with Positive real part, and the system is not stable.

Special case: If a first column term in any row is zero but the remaining terms are not, the zero term is replaced by a very small positive number (ε) and the rest of the array is evaluated. For example, consider the following equation: $S^3 + 2S^2 + S + 2 = 0$

The array of the coefficients is:

$$\begin{array}{lll} s^3 & 1 & 1 \\ s^2 & 2 & 2 \\ s^1 & 0 \approx \epsilon & \\ s^0 & 2 & \end{array}$$

If the sign of the coefficient above the zero is the same as that below it, it indicates that there is a pair of imaginary roots. for the last equation, we have two roots at $s = \pm j$.

If the sign of the coefficient above the zero is opposite of that below it, it indicates that there is one sign change.

for example: $s^3 - 3s + 2 = (s-1)^2 (s+2) = 0$

The array of coefficients is:

	s^3	1	-3
one sign change	s^2	$0 \approx \epsilon$	2
	s^1	$-3 - \frac{2}{\epsilon}$	
one sign change	s^0	2	

If all coefficients in any derived row are zero, it indicates that there are roots of equal magnitude lying radially opposite in the s-plane. In such case, the evaluation of the rest of the array can be continued by forming an auxiliary polynomial in the next row, and such roots can be found by solving the auxiliary polynomial.

example: Consider the following equation:

$$S^5 + 2S^4 + 24S^3 + 48S^2 - 25S - 50 = 0$$

$\Rightarrow$ S^5	1	24	-25
S^4	2	48	-50 ← Auxiliary polynomial P(s)
S^3	0	0	

$$P(s) = 2S^4 + 48S^2 - 50$$

which indicates that there are two pairs of roots of equal magnitude and opposite sign

Solving P(s) by equating to zero and find the derivative with respect to (S)

$$P(s) = 2S^4 + 48S^2 - 50 = 0$$

$$\rightarrow \frac{dP(s)}{dS} = 8S^3 + 96S \quad \text{-----} (1)$$

→ becomes:

S^5	1	24	-25
S^4	2	48	-50
S^3	8	96 ← coefficients of $\frac{dP(s)}{dS}$	
S^2	24	-50	
S^1	112.7	0	
S^0	-50		

We see that there is one change in sign in the first column of the new array. Thus, the original equation has one root with a positive real part.

now, solving the auxiliary polynomial

$$2S^4 + 48S^2 - 50 = 0$$

$$S^2 = 1 \quad , S^2 = -25$$

or $\quad S = \mp 1 \quad , S = \mp J5$

example: Consider a system with the following closed-loop transfer function:

$$\frac{C(S)}{R(S)} = \frac{K}{S(S^2+S+1)(S+2)+K}$$

find the values of K that ensure the stability of the system.

sol. $\quad S^4 + 3S^3 + 3S^2 + 2S + K = 0$

S^4	1	3	K
S^3	3	2	0
S^2	$\frac{7}{3}$	K	
S^1	$2 - \frac{9}{7}K$		
S^0	K		

for stability K must be positive, and all Coefficients in the first column must be positive.

$$\rightarrow 2 - \frac{9}{7}K > 0 \quad , \text{and} \quad K > 0$$

$$2 > \frac{9}{7}K \rightarrow \frac{14}{9} > K$$

$$\therefore \quad 0 < K < \frac{14}{9}$$

example: Determine if the following characteristic equation has any roots with positive real parts:

$$s^4 + s^3 - s - 1 = 0$$

sol. Note that the s^2 term is zero

s^4	1	0	-1
s^3	1	-1	0
s^2	1	-1	$\rightarrow P(s) = s^2 - 1 \rightarrow \frac{dP(s)}{ds} = 2s - 0 = 0$
s^1	$\cancel{0}$ 2	$\cancel{0}$ 0	
s^0	-1		

one change in sign

We have one change in sign, therefore the characteristic equation has one root with a positive real part.

$$P(s) = s^2 - 1 = 0 \rightarrow s^2 = 1 \rightarrow s = \mp 1$$

$\rightarrow$ the positive root is $+1$.

example: The characteristic equation of a given system is: $s^4 + 6s^3 + 11s^2 + 6s + K = 0$

what restrictions must be placed up on the parameter K in order to ensure that the system is stable?

sol. $$s^4 + 6s^3 + 11s^2 + 6s + K = 0$$

s^4	1	11	K
s^3	6	6	0
s^2	10	K	0
s^1	$\frac{60-6K}{10}$	0	
s^0	K		

for stability, $K > 0$ and $\frac{60-6K}{10} > 0$

$\rightarrow K < 10$

$\rightarrow \quad 0 < K < 10$

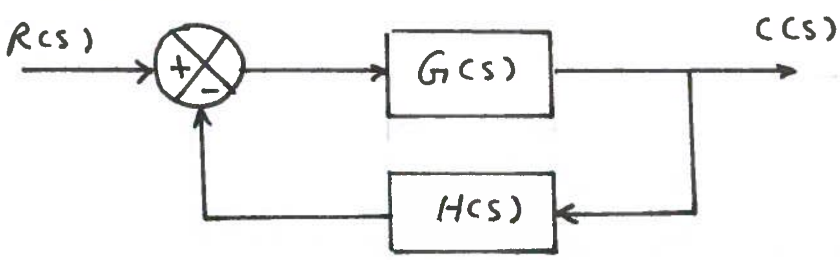
Root Locus method:

The performance of a feedback system can be described in terms of the location of the roots of the characteristic equation in the s-plane. Root locus plot is a graph showing how the roots of the characteristic equation move around the s-plane as a single parameter varies, typically the open loop gain K.

In designing a linear control system, we find that the root locus method is quite useful since it indicates the manner in which the open loop poles and zeros should be modified so that the response meets system performance specifications.

Angle and Magnitude conditions:-

Consider the system shown:



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

The characteristic equation is

$$1 + G(s)H(s) = 0 \qquad \text{or} \qquad G(s)H(s) = -1$$

and as $G(s)H(s)$ is complex quantity

→ angle condition, $\angle G(s)H(s) = \pm 180^\circ (2k+1)$, $(k = 0, 1, 2, \dots)$

magnitude condition, $|G(s)H(s)| = 1$

The values of (s) that fulfill both the angle and magnitude conditions are the roots of the characteristic equation, or the closed loop poles.

In general, the characteristic equation can be written as:

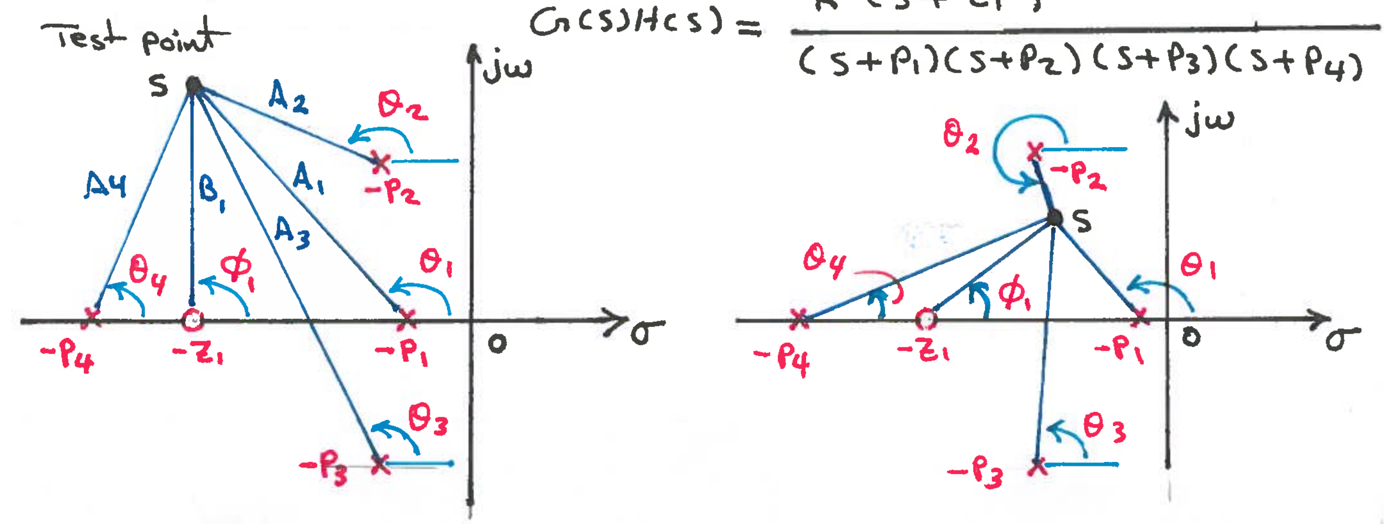
$$1 + \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = 0$$

then the root loci of the system are the loci of the closed loop poles as the gain K is varied from zero to infinity.

To sketch the root loci of a system we must know the location of the poles and zeros of $G(s)H(s)$.

*The angles of the complex quantities originating from the open loop poles and zeros to the test point (s) are measured in the counterclockwise direction, for example:

$$G(s)H(s) = \frac{K(s+z_1)}{(s+p_1)(s+p_2)(s+p_3)(s+p_4)}$$



where $-P_2$ and $-P_3$ are complex conjugate poles,

$$\rightarrow \underline{/G(s)H(s)} = \phi_1 - \theta_1 - \theta_2 - \theta_3 - \theta_4$$

and $$|G(s)H(s)| = \frac{K B_1}{A_1 A_2 A_3 A_4}$$

where A_1, A_2, A_3, A_4, B_1 are the magnitude quantities of the complex poles and zeros in $G(s)H(s)$,

Note: The roote loci are always symetrical with respect to the axis.

Sketching the Root locus:-

General Rules:

Rule 1: obtain the open loop poles and zeros of $G(s)H(s)$,

The number of root locus branches should be equal to the number of the open loop poles.

Rule 2: Each locus starts from open loop poles when $K=0$ and terminates at open loop zeros (finite zeros or infinite means zeros at ∞), when $K=\infty$.

Rule 3: Root loci either move along the real axis or can happen to be as complex conjugate.

Rule 4: A value of "S" on the real axis is a point on the root locus if the total number of poles and zeros on the real axis to the right of the point is odd.

Rule 5 : As one moves far enough away from the open loop poles and zeros, the root loci become asymptotic to lines making angles (θ_a) to the real axis,

$$\text{angles of asymptotes} = \theta_a = \frac{(2i+1)\pi}{P-Z}$$

where: P is the number of open loop poles

Z is the number of open loop zeros

$i = \mp (0, 1, 2, \dots)$

Rule 6 : The intersection of asymptotes with real axis at point σ_a is given by:

$$\sigma_a = \frac{\sum \text{Poles} - \sum \text{zeros}}{P - Z}$$

Rule 7: Intersection of the root loci with the imaginary axis can be easily obtained by use of Routh Criterion or by putting $S = j\omega$ and equating imaginary parts to zero to find ω, the intersection.

Rule 8: The break away point from the real axis is obtained by differentiating the open loop transfer function with respect to (S) and equating it to zero.

Rule 9: Angles of departure and arrival of a root loci can be determined from the complex poles and zeros respectively and is given by:

$$\theta_d = 180^\circ - \phi_P + \phi_Z$$

ϕ_p : sum of all the angles subtended by all other poles.

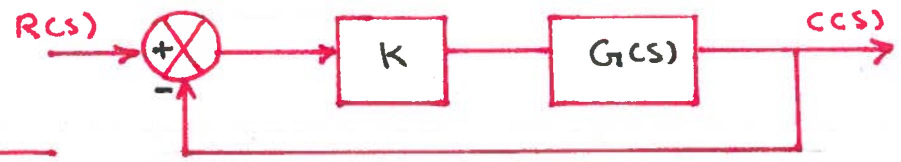
ϕ_z : sum of all angles of any zeros.

Note: Regarding breakaway and break-in points (Rule 8), are the points at which the closed loop poles leave the real axis and convert to complex form. The break-in points are vice versa, denoting them by (σ):

$$\sum_{i=1}^{m} \frac{1}{\sigma + z_i} = \sum_{i=1}^{n} \frac{1}{\sigma + p_i}$$

where z_i and p_i are the negative of the zeros and poles of $G(s)H(s)$, and σ has real values only (complex values are not acceptable).

example: Draw the location of poles of the closed loop system shown as K varies.



$G(s) = \frac{1}{s(s+4)}$

Sol. from rules

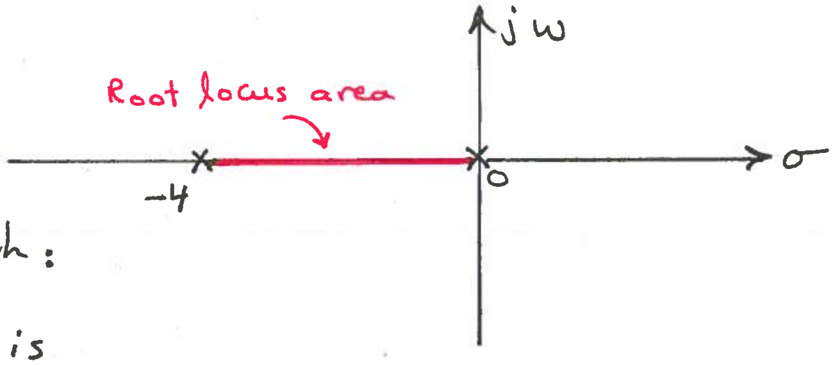
we have 2 branches because of having 2 open loop poles at $s = 0, -4$ and go to open loop zeros at $s = \infty, \infty$

Asymptotes : $\sigma_a = \frac{(0 + (-4)) - (0)}{2 - 0} = -2$

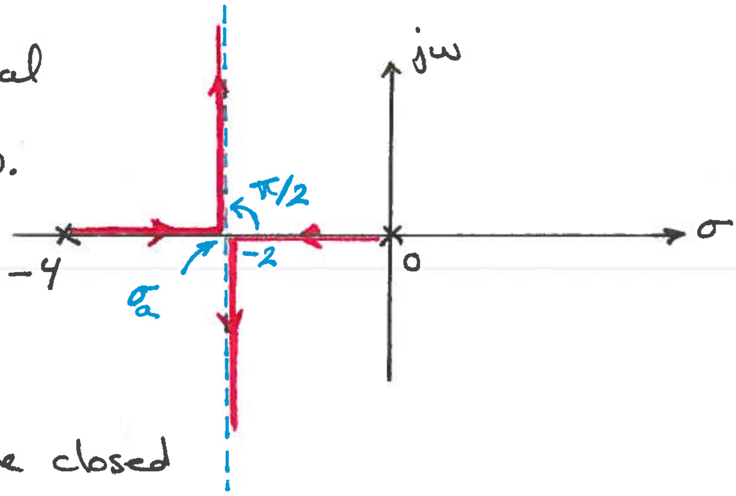
angles of asymptotes : $\theta_a = \frac{(2i+1)}{2-0}\pi = \frac{\pi}{2}, -\frac{\pi}{2}$

any other i, same results we get

note that the real axis between 0 and 4 is a part of the root locus (at the left of odd number of poles and zeros).



now we can sketch:



note Root locus is symetric about the real axis and cannot overlap. and the direction of the arrows shows the movement of the closed loop poles as K increases.

we can see, if K is small, the system is overdamped and as K increases, the system will be underdamped with oscillating response.

breakaway point is:

$$\frac{1}{\sigma+\infty} = \frac{1}{\sigma+0} + \frac{1}{\sigma+4}$$

$$\longrightarrow -\frac{1}{\sigma} = \frac{1}{\sigma+4} \Rightarrow \sigma = \underline{\underline{-2}}$$

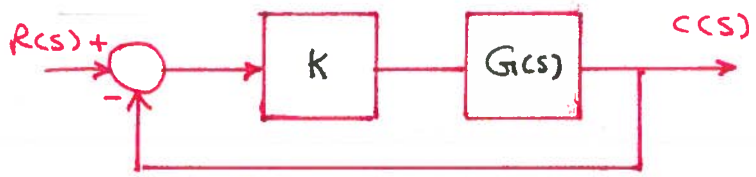
or we have $G(s)H(s) = \dfrac{1}{s(s+4)}$

$$\longrightarrow \frac{d}{ds} G(s)H(s) = \frac{0-1[s+s+4]}{s^2(s+4)^2} = 0$$

$$\longrightarrow 2s = -4 \longrightarrow s = \underline{\underline{-2}}$$

example: Draw the location of poles of the system shown as K varies.

$$G(s) = \frac{10}{s(s+5)(s+10)}$$



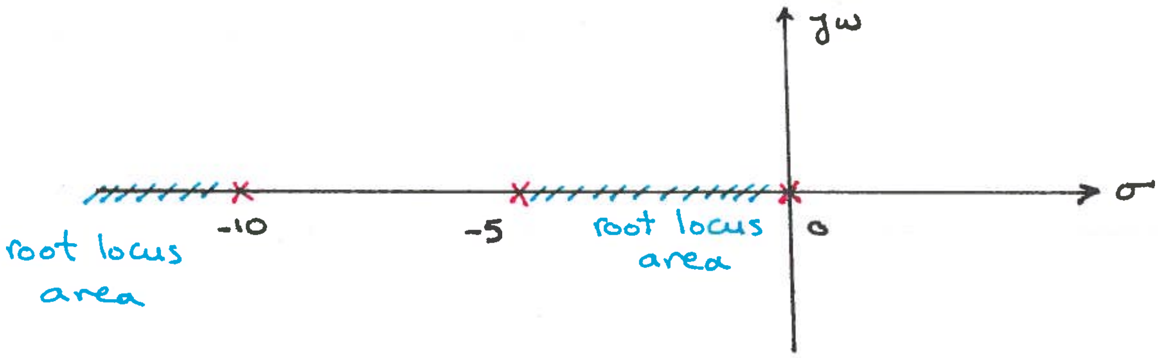
Sol

number of branches are 3, start at open loop poles $s = 0, -5, -10$ & end at open loop zeros $s = \infty, \infty, \infty$.

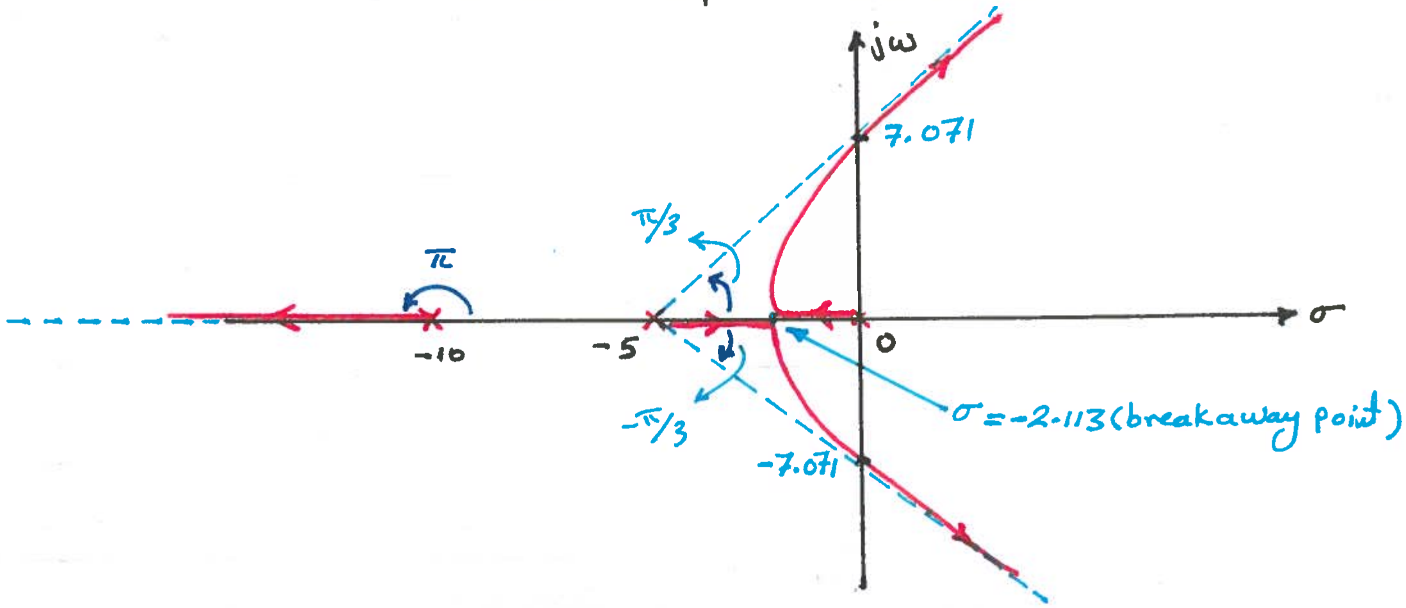
Asymptotes: $\sigma_a = \dfrac{(0-10-5)-(0)}{3-0} = -5$

Angles of asymptotes: $\theta_a = \dfrac{(2i+1)\pi}{3-0} = \dfrac{\pi}{3}, \pi, -\dfrac{\pi}{3}$

(we have 3 different asymptotes)



root locus converges to asymptotes as K increases and go to the right half plane, then system becomes unstable as K further increases after oscillations.



breakaway point :

$$\frac{1}{\sigma+0} + \frac{1}{\sigma+5} + \frac{1}{\sigma+10} = 0$$

$$\rightarrow 3\sigma^2 + 30\sigma + 50 = 0$$

$$\rightarrow \sigma_1 = -2.113 \quad \& \quad \sigma_2 = \underline{-7.886} \rightarrow \text{not valid}$$

$\therefore$ breakaway Point is -2.113

Imaginary crossing:

$$T.F = \frac{\frac{10K}{S(S+5)(S+10)}}{1 + \frac{10K}{S(S+5)(S+10)}}$$

$$T.F = \frac{10K}{S^3 + 15S^2 + 50S + 10K}$$

using Routh table:

S^3	1	50	
S^2	15	10K	$\Rightarrow K>0$, and $K<75$
S^1	$\frac{750-10K}{15}$		for stability.
S^0	10K		or $0 < K < 75$

The value of K that makes the S^1 term Zero is 75

→ The crossing points on imaginary axis can be found by solving the auxiliary equation obtained from the S^2 row,

$$\rightarrow 15S^2 + 10K = 15S^2 + 750 = 0 \Rightarrow S = \pm j\,7.071$$

hence, as K increases further, the system becomes unstable.

another way to find the crossing points:

Let $s = j\omega$ in the characteristic equation

$\rightarrow (j\omega)^3 + 15(j\omega)^2 + 50(j\omega) + 10K = 0$

$\rightarrow \omega = 7.071$

example: Given that $G(s) = \frac{K(s+3)}{s(s-3)}$ for the unity feedback control system. Find K corresponding to imaginary axis crossing and the corresponding poles.

sol: number of branches are 2, start at $S = 0, 3$ and end at $S = -3, \infty$.

Asymptotes, $\sigma_a = 6$

Angles of asymptote: $\theta_a = \pi$

Transfer function: $T.F = \dfrac{K(S+3)}{S^2 - 3S + KS + 3K}$

S^2	1	3K
S^1	K−3	0
S^0	3K	

if $K=3$ then we will have row of zero. For K smaller than 3, we endup with unstable system. For $K>3$, no sign change.

And at certain K we are at the axis, $K=3$.

and for $K=3$, from S^2 row we get:

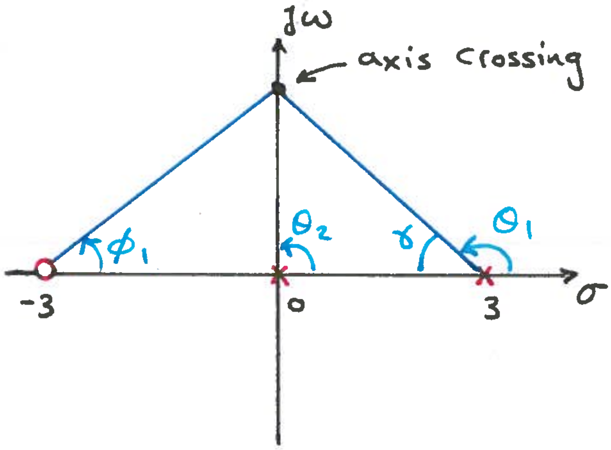
$$S^2 + 3K = 0 \rightarrow S = \mp j3$$

another way: from characteristic equation, put $S = j\omega$

then $(j\omega)^2 - 3j\omega + j\omega K + 3K = 0$

for $K=3 \rightarrow \omega = \mp 3$

another way : From angle Criterion



$\theta_1 + \theta_2 - \phi_1 = 180^\circ$

$\rightarrow \theta_1 + 90 - \phi_1 = 180^\circ$ ---- (1)

but $\theta_1 + \gamma = 180^\circ$

and note that $\gamma = \phi_1$

$\rightarrow \theta_1 + \phi_1 = 180^\circ$ --- (2)

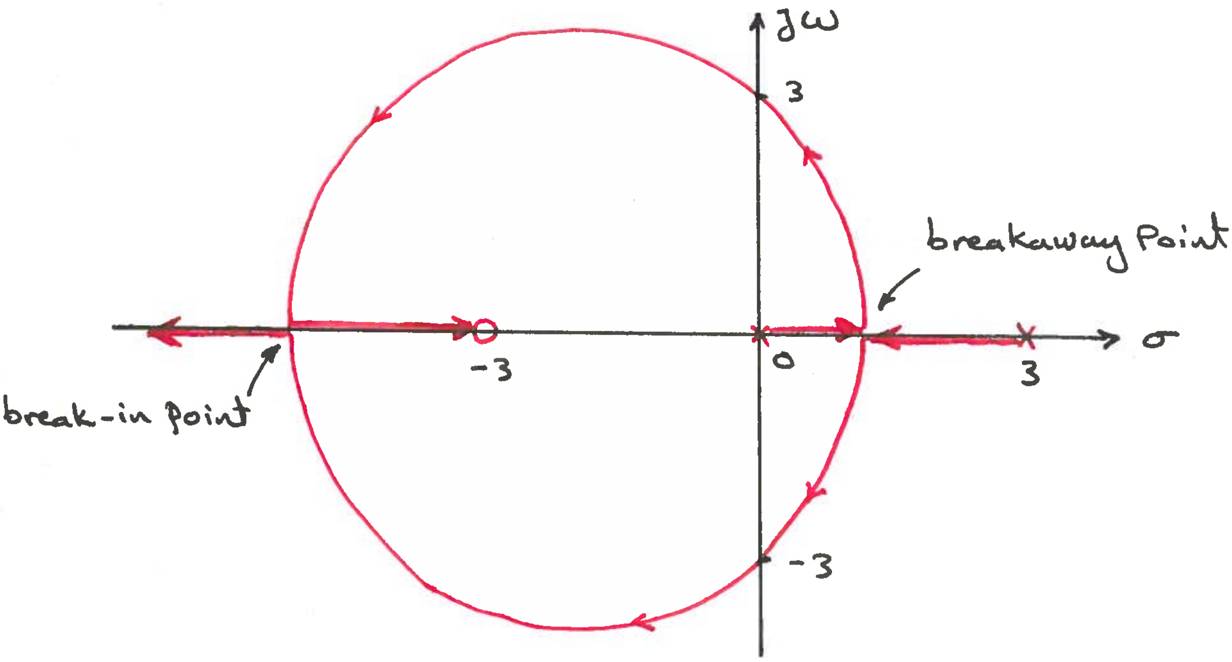
$\rightarrow \phi_1 = 45^\circ \longrightarrow \sigma = 3$

and then the imaginary axis Crossing is $\pm j\omega = \pm j3$

now we need to find the corresponding K

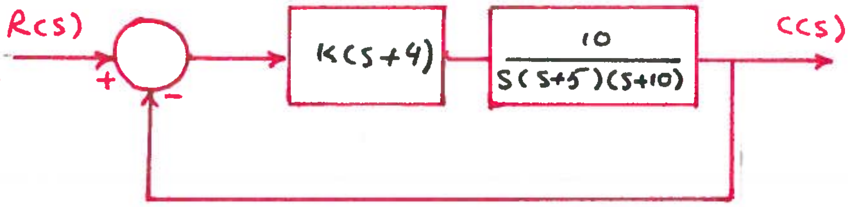
from magnitude Criterion,

$$K\,|G(s)H(s)| = 1 = \left| \frac{K\sqrt{\omega^2+9}}{\omega\sqrt{\omega^2+9}} \right|_{\omega=3} = 1 \rightarrow K = 3$$



example:

Draw the Root locus and find the breakaway/break-in points.



Sol: number of branches = 3, start at $s = 0, -5, -10$

and end at $s = -4, \infty, \infty$

Asymptote: $\sigma_a = -5.5$

Angles of asymptote: $\theta_a = \frac{\pi}{2}, -\frac{\pi}{2}$

The breakaway point:

$$\frac{1}{\sigma+4} = \frac{1}{\sigma+0} + \frac{1}{\sigma+5} + \frac{1}{\sigma+10} \rightarrow \sigma = -6.4$$