AL-MAMON UNIVERSITY COLLAGE

DEPARTMENT OF ELECTRICAL POWER ENGINEERING TECHNIQUES

Lecture notes 3

Mechanical Vibrations

Part I

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Basic Concepts of Mechanical Vibrations

This lecture notes consist the following topics:

- Damped Single Degree-of-Freedom
- Viscous Damping
- Equation of Motion
- Homogeneous solution of the Equation of Motion
- Cases of Damped

3.1 Damped Single Degree-of-Freedom

We have seen in the previous lecture that the simple oscillator under idealized conditions of no damping, once excited, will oscillate indefinitely with a constant amplitude at its natural frequency. Experience indicates, however, that it is not possible to have a device which vibrates under these ideal conditions.

الحركة الاهتزازية في ظل ظروف مثالية من عدم التخميد سوف تستمر الى مالانهاية مع سعة ثابتة عند تردده الطبيعي ولكن ،في الحقيقة تشير التجربة إلى أنه لا يمكن الحصول على جهاز يهتز في ظل هذه الظروف المثالية.

Forces designated as frictional or damping forces are always present in any physical system undergoing motion. These forces dissipate energy more precisely, the unavoidable presence of these frictional forces constitutes a mechanism through which the mechanical energy of the system, kinetic or potential energy, is transformed to other forms of energy such as heat.

حيث في الانظمة المهتزة الحقيقية دائمًا ما تكون القوى المأثرة كقوى احتكاك أو مخمدات موجودة في أي نظام فيزيائي يخضع للحركة. تعمل هذه القوى على تبديد الطاقة بشكل أكثر دقة ، ويشكل وجود قوى الاحتكاك التي من خلالها يتم تحويل الطاقة الميكانيكية للنظام ، الطاقة الحركية أو الكامنة ، إلى أشكال أخرى من الطاقة مثل الحرارة او صوت.

3.2 Viscous Damping

In considering damping forces in the dynamic analysis of structures, it is usually assumed that these. forces are proportional to the magnitude of the velocity, and opposite to the direction of motion. This type of damping is known as viscous damping; it is the type of damping force that could be developed in a body restrained in its motion by a surrounding viscous fluid.

عند النظر في قوى التخميد في التحليل الديناميكي للهياكل ، يُفترض عادةً أنها. تتناسب القوى مع مقدار السرعة ومعاكسة لاتجاه الحركة. يُعرف هذا النوع من التخميد باسم التخميد اللزج ؛ إنه نوع من قوة التخميد التي يمكن تطويرها في جسم مقيّد في حركته بواسطة سائل لزج محيط.

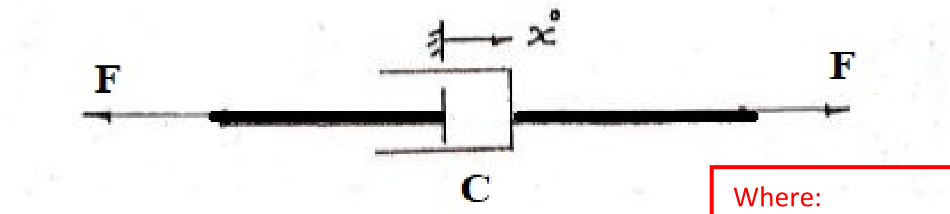

Fig 3.1 Viscous Damper

Where:

F: Force (N) القوة

C: damping coefficient (Ns/m) معامل المضائلة (التخمد)

3.3 Equation of Motion

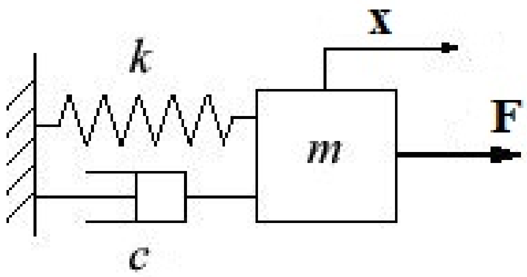

Fig 3.2 Damped Single Degree-of-Freedom System

SDOF vibration can be analyzed by Newton's second law of motion,

$$F = m*a.$$

The analysis can be easily visualized with the aid of a free body diagram.

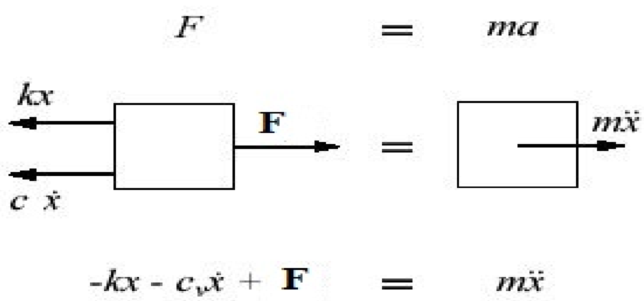

Fig 3.2 Free Body Diagram

$$\sum F = m\ddot{x}$$

$$-kx - c\dot{x} = m\ddot{x}$$

$$\therefore m\ddot{x} + c\dot{x} + kx = 0 \qquad (10)$$

3.3.1 Homogeneous solution of the Equation of Motion:

The equation of motion for free damped single degree of freedom system is an ordinary differential equation of the second order. It is called a homogeneous differential equation because the right-hand side of the equation is equal to zero. To derive the homogeneous solution, first assume that the solution is in the form of:

$$x(t) = Ae^{st} \qquad (11)$$

Differentiating Eq. (11) to get the first derivative:

$$\dot{x}(t) = Ase^{st} \qquad (12)$$

and second derivative:

$$\ddot{x}(t) = As^2e^{st} \qquad (13)$$

Substituting Eq. (13), Eq (12) and Eq. (11) in Eq. (10) we get:

$$mAs^2e^{st} + cAse^{st} + kAe^{st} = 0$$

$$Ae^{st}(ms^2 + cs + k) = 0$$

$$\because Ae^{st} \neq 0 \rightarrow (ms^2 + cs + k) = 0$$

$$\therefore s^2 + \frac{c}{m}s + \frac{k}{m} = 0 \qquad (14)$$

Now, consider ζ as the damping ratio and it is defined as:

Damping ratio *is the ratio between the damping coefficient of a system and the critical damping coefficient*.

نسبة التخميد هي النسبة بين معامل التخميد للنظام ومعامل التخميد الحرج

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2m\omega_n} \qquad (15)$$

Where:

$$c_{cr} = 2m\omega_n$$

is the **Critical Damping Coefficient** and can be defined as the minimum damping required to allow a mass to return to its original position in the shortest time without vibration.

معامل التخميد الحرج ويمكن تعريفه
على أنه الحد الأدنى للتخميد المطلوب
للسماح للكتلة بالعودة إلى موقعها
الأصلي في أقصر وقت بدون اهتزاز.

We get from Eq (15):

$\frac{c}{m} = 2\zeta\omega_n$ and putting $\frac{k}{m} = \omega_n^2$ in Eq (14):

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \qquad (16)$$

which is a quadratic equation of the form ($as^2 + bs + c = 0$) with two roots that can be solved using the following formula:

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Therefore, Eq. (16) has the following roots:

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \qquad (17)$$

Hence, the exact solution of the equation of motion will be:

$$x(t) = Ae^{s_1 t} + Be^{s_2 t} \qquad (18)$$

There are three cases for the roots s_1 and s_2 depending on the value of the damping ratio (ζ):

$\sqrt{\zeta^2 - 1}$

- → +ve (if $\zeta > 1$)
- → zero (if $\zeta = 1$)
- → -ve (if $\zeta < 1$)

<u>Case 1: Over-damped system ($\zeta > 1$)</u>:

$$x(t) = A e^{(-\zeta \omega_n - \omega_n \sqrt{(\zeta^2 - 1)})t} + B e^{(-\zeta \omega_n + \omega_n \sqrt{(\zeta^2 - 1)})t} \qquad (19)$$

When $\zeta > \mathbf{1}$ the two roots are real

Figure 3.3 shows the plot of equation (19) for different cases of over-damping. It is shown that the mass will return to its original position without oscillation. The increase of damping factor ζ will increase the time needed to reach the position of rest.

عندما تكون $\zeta > \mathbf{1}$ فان جذري المعادلة يكونان عددين حقيقين وفي هذه الحالة يقال ان المضألة (التخميد) مرتفعة ولاتؤدي الى تأرجح الكتلة.

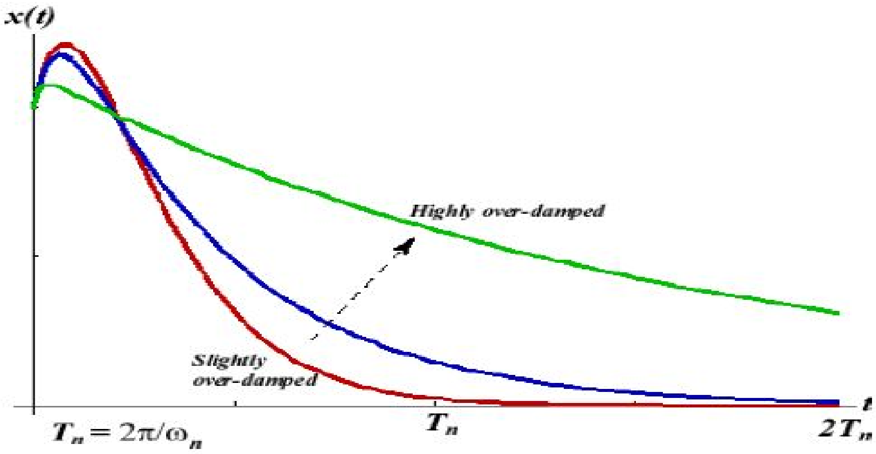

Fig 3.3 Response of over-damped system

Case 2: Critically-damped system (ζ = 1):

When **ζ = 1** the two roots are real

عندما **1 = ζ** فان جذري المعادلة يكونان متساويين ولايحدث تأرجح وتسمى الحالة الحرجة

$$s_1 = s_2 = -\zeta\,\omega_n = -\omega_n \longrightarrow \text{Two real equal roots}$$

Therefore, the response of critically-damped system will be:

Figure 3.4 shows the plot of Eq. (20).

$$x(t) = (A + Bt)e^{-\omega_n t} \qquad (20)$$

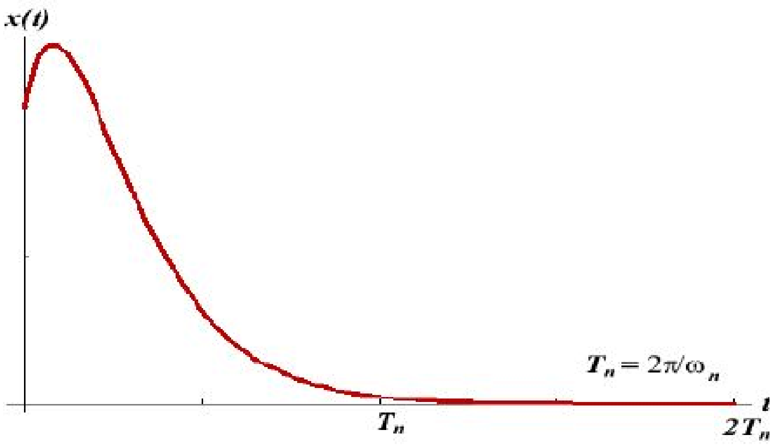

Fig. 3.4 Response of critically-damped system

Case 3: Under-damped system ($\zeta < 1$):

When $\zeta < 1$ the two roots are imaginary

عندما تكون $\zeta < 1$ فان جذري المعادلة يكونان عددين مركبين وفي هذه الحالة تكون المضألة منخفضة ويؤدي الى تأرجح الكتلة.

$$s = -\zeta\,\omega_n \pm i\,\omega_n\sqrt{1-\zeta^2} \longrightarrow \text{Two imaginary roots}$$

where $i = \sqrt{-1}$

Now, we will define the damped natural frequency:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \qquad (21)$$

Therefore, the roots can be written as follows:

$$s_1 = -\zeta\,\omega_n - i\,\omega_d \ \text{ and } \ s_1 = -\zeta\,\omega_n + i\,\omega_d$$

Substituting in Eq. (18) we get:

$$x(t) = A\,e^{(-\zeta\,\omega_n - i\omega_d)t} + B\,e^{(-\zeta\,\omega_n + i\omega_d)t}$$

$$= A\,e^{(-\zeta\,\omega_n t)}\,e^{(-i\omega_d\,t)} + B\,e^{(-\zeta\,\omega_n t)}\,e^{(i\omega_d\,t)}$$

$$= e^{(-\zeta\,\omega_n t)}\left[A\,e^{(-i\omega_d\,t)} + B\,e^{(i\omega_d\,t)}\right]$$

Recalling that: $e^{i\,\omega_d} = \cos\omega_d\,t + i\sin\omega_d\,t$ *and* $e^{-i\,\omega_d} = \cos\omega_d\,t - i\sin\omega_d\,t$

Then, the response of under-damped system can be written in trigonometric form as follows:

$$x(t) = e^{-\zeta\,\omega_n t}\left[A_1\cos\omega_d\,t + B_1\sin\omega_d\,t\right]$$
$$or \quad = C\,e^{-\zeta\,\omega_n t}\sin(\omega_d\,t + \varphi_1) \qquad \textbf{(22)}$$

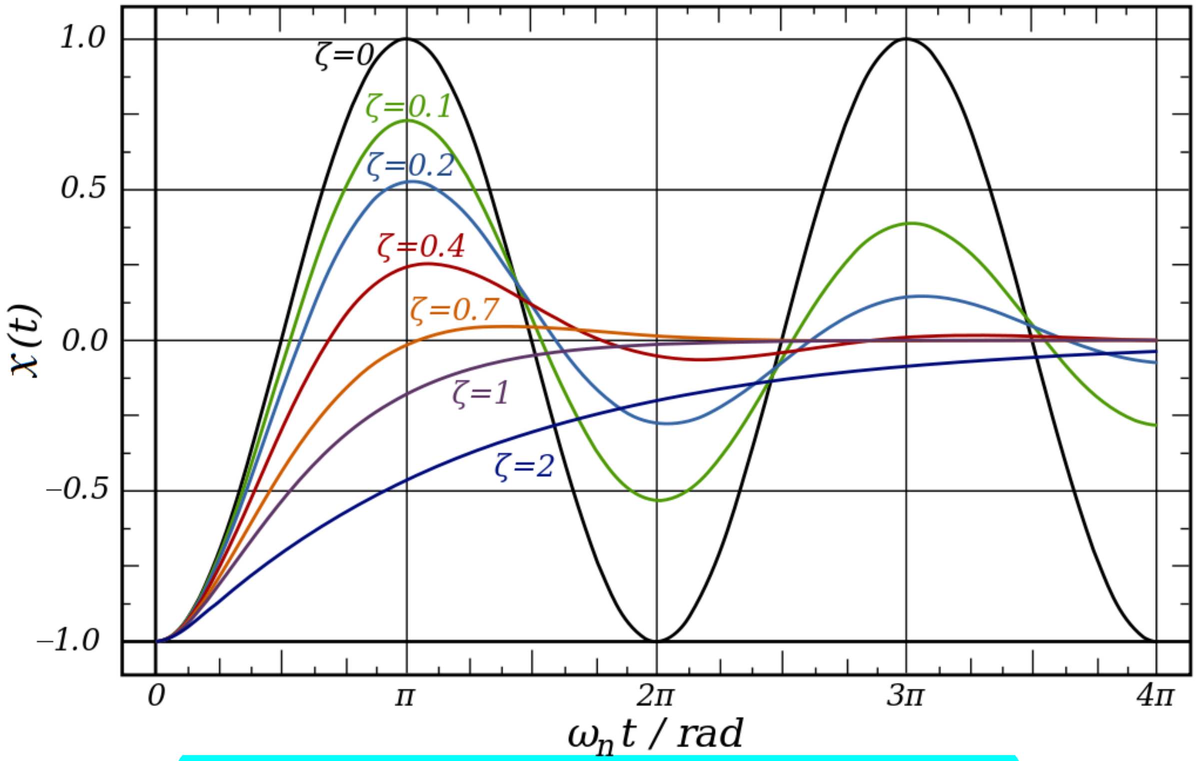

Fig. 3.5 The effect of varying damping ratio on a damped system

Figure 3.3 shows the effect of varying damping ratio from $\zeta = 0$ to $\zeta = 2$ on a damped system.

Example 7: - A single degree of freedom spring-mass-damper system has a mass (60 kg) and spring stiffness (k = 6000 N/m). Determine the following:

a) The **Critical Damping Coefficient**.
b) The **Damped Natural Frequency** when $C = \frac{c_{cr}}{3}$

Solution: -

$$c_{cr} = 2m\omega_n$$

Where $$\omega_n = \sqrt{\frac{k}{m}}$$

$$\omega_n = \sqrt{\frac{6000}{60}} = 10\ rad/s$$

$$c_{cr} = 2 * 60 * 10 = 1200\ N.s/m$$

$$C = \frac{c_{cr}}{3} = \frac{1200}{3} = 400 \text{ N.s/m}$$

$$\zeta = \frac{c}{c_{cr}} = \frac{400}{1200} = 0.333$$

$$\omega_D = \omega_n \sqrt{1 - \zeta^2}$$

$$\omega_D = 10 \sqrt{1 - 0.333^2}$$

$$\omega_D = 9.428\, rad/s$$