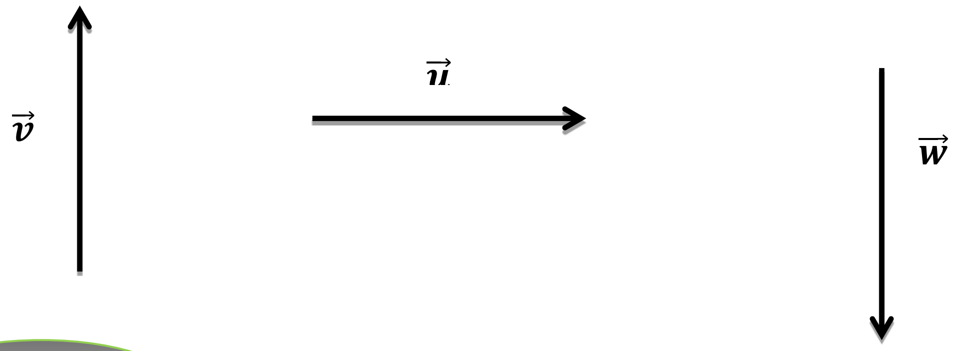
Vectors

Definition **Scalar Quantities** A scalar is a quantity which has only a magnitude in space. Such as: length, weight, volume, Temperature… etc..

Definition **Vector Quantities** are a quantity which has both Magnitude and direction in space such as: force, speed….. etc

$\vec{v}= \overrightarrow{AB}$, $\vec{u}=\overrightarrow{CD}$,$\vec{w}= \overrightarrow{EF}$



Note

(1) The zero vector is just a point, and it is denoted by 0, and has arbitrary direction ,which is written as $\vec{0}$

(2) Given the two points $A = (a_0, b_0, c_0)$ and $B = (a_1, b_1, c_1)$ the vector with the representation $\overrightarrow{AB} = (a_1 - a_0, b_1 - b_0, c_1 - c_0)$

Note that the vector above is the vector that starts at A and ends at B.

The vector that starts at B and ends at A is $\overrightarrow{BA} = (a_0 - a_1,\ b_0 - b_1, c_0 - c_1)$

(3) $\overrightarrow{AB} = -\overrightarrow{BA}$

Example 1: Given the vector for each of the following.

1- The vector from A=(2,-7,0) to B= (1,-3,-5)

2- The vector from C=(1,-3,-5) to D= (2,-7,0)

Solution

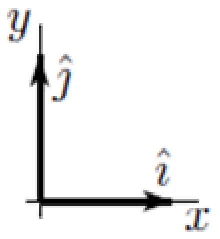
1- $\overrightarrow{AB} = (1-2, -3+7, -5-0) = (-1, 4, -5)$

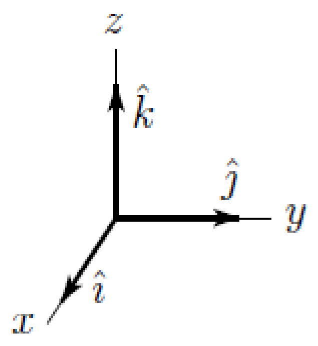
2- $\overrightarrow{CD} = (2-1, -7+3, 0+5) = (1, 4, 5)$

Definition **Unit vector** A vector of length 1 is called a *unit vector*. In an *xy*-coordinate system the unit vectors along the *x*- and *y*-axis are denoted by i and j, respectively. In an *xyz*-coordinate system the unit vectors along the *x*-, *y*- and *z*-axis are denoted by i, j and k, respectively. Thus:

$\vec{i}$ = (1,0) , j = (0,1) (2- dimension)

$\vec{i}$ = (1,0,0) , j = (0,1,0), $\vec{k}$ = (0,0,1) (3- dimension)





Note

All vectors can be expressed as linear combinations of the unit vectors

$\vec{v}$ = (v_1 , v_2)= v_1i + v_2j (2- dimension)

$\vec{v}$ = (v_1 , v_2 , v_3)= v_1i + v_2j + v_3k (3- dimension)

Example Write the vector $\vec{u}$ = (3, 4,5) as linear combinations of the unit vector

Solution

$\vec{u}$ = 3i + 4j + 5k

Definition magnitude or (length) The magnitude or (length) of the vector $v = (v_1, v_2, v_3)$ is denoted by the symbol$\|\vec{v}\|$ or $|\vec{v}|$ is.

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Properties of the magnitude

a) $\vec{v} . \vec{v} = \| \vec{v}\|^2$

b) $\| \vec{0}\| = 0$

c) $\| \vec{v}\| \geq 0$

d) $\| \vec{v}\|$ =0 if and only if $\vec{v} = \vec{0}$

e) $\| \vec{v} + \vec{u}\| \leq \| \vec{v}\| + \| \vec{u}\|$

f) $\| \vec{v}.\vec{u}\| \leq \| \vec{v}\| . \| \vec{u}\|$

Example :

Find the length of the vector , $\vec{v}$=(-1 , 3 , 2)

Solution

$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(-1)^2 + (3)^2 + (2)^2} = \sqrt{1+9+4} = \sqrt{14}$

Note

The magnitude of a vector is *not* in general equal to the sum of the magnitudes of the two original vectors.

For Example

The magnitude of the vector (3, 0, 0) is 3, and the magnitude of the vector (-2, 0, 0) is 2,
but the magnitude of the vector(3, 0, 0) + (-2, 0, 0) is 1, not 5!

A unit vector (u_n) for any vector $\vec{A}$

Defined as a vector whose magnitude is unity and is along of $\vec{A}$ that is

$$u_n = \frac{\vec{A}}{|\vec{A}|}$$

Example:

Find the vector $\vec{A}$ directed from point (2,-4,1) to point(0,-2,0) in Cartesian coordinates and find the unit vector along $\vec{A}$

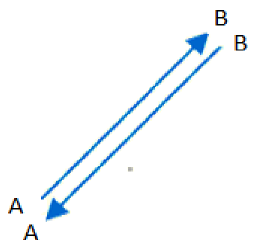
Solution

$$\vec{A} = -2i+2j-k$$

$$|\vec{A}| = \sqrt{(-2)^2 + (2)^2 + (-1)^2} = 3$$

$$u_n = \frac{\vec{A}}{|\vec{A}|} = \frac{-2i+2j-k}{3} = \frac{-2}{3}i + \frac{2}{3}j - \frac{1}{3}k$$

Definition **Inverse Vectors** **An inverse vector is a vector of equal magnitude to the original but in the opposite direction**



- $\overrightarrow{AB} = -\overrightarrow{BA}$
- $\overrightarrow{AB} + \overrightarrow{BA} = 0$

Vector Arithmetic

1- addition of the vector:

Given the two vectors $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$
The addition of the two vectors is given by the following formula

$$\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$$

For $\vec{v} = a_1 i + a_2 j + a_3 k$ and $\vec{w} = b_1 i + b_2 j + b_3 k$ be two vectors
Then $\vec{v} + \vec{w} = (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k$

2- Subtraction of vector:

Given the two vectors $\vec{v} = (v_1, v_2, v_3)$ and $\vec{u} = (u_1, u_2, u_3)$
the Subtraction vector is

$$\vec{v} - \vec{u} = (v_1 - u_1, v_2 - u_2, v_3 - u_3)$$

For $\vec{v} = a_1 i + a_2 j + a_3 k$ and $\vec{w} = b_1 i + b_2 j + b_3 k$ be two vectors
Then $\vec{v} - \vec{w} = (a_1 - b_1)i + (a_2 - b_2)j + (a_3 - b_3)k$
The subtraction operation between two vectors $\vec{u} - \vec{v}$ can be understood as a vector addition between the first vector and the opposite of the second vector:

Definition scalar multiplication

Given the vectors $\vec{v} = (v_1, v_2, v_3)$ and any number C scalar multiplication is

$$C\,\vec{v} = (Cv_1, Cv_2, Cv_3)$$

For $\vec{v} = v_1 i + v_2 j + v_3 k$

Then $C\,\vec{v} = Cv_1 i + Cv_2 j + Cv_3 k$

Scalar multiplication obeys the following rules :

- Distributive in the scalar: $(k + d)\vec{v} = k\vec{v} + d\vec{v}$
- Distributive in the vector: $k(\vec{v} + \vec{w}) = k\vec{v} + k\vec{w}$
- Associate of product of scalars with scalar multiplication: $(kd)\vec{v} = k(d\vec{v})$
- Multiplying by 1 does not change a vector: $1\vec{v} = \vec{v}$
- Multiplying by 0 gives the zero vector: $0\vec{v} = \vec{0}$
- Multiplying by −1 gives the additive inverse: $(-1)\vec{v} = -\vec{v}$

Properties of vector algebra

If $\vec{v}$, $\vec{w}$ and $\vec{u}$ are vectors and a and b two numbers then we have the following properties:

a. For any vector v there is a vector (−v) such that $\vec{v} + (-\vec{v}) = \vec{0}$

b. $\vec{v}+\vec{w}=\vec{w}+\vec{v}$ Commutative Law

c. $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ Associative Law

d. $\vec{v}+0 = \vec{v} = 0+\vec{v}$ Additive Identity

e. $(a + b)\vec{v} = a\vec{v} + b\vec{v}$

Example

3) Find $\vec{u} + \vec{v}$ for $\vec{u} = (3i + 4j)$, $\vec{v} = (-2i + j)$.

4) Find $\vec{u} - \vec{v}$ for $\vec{u} = 5(3i + 2j)$, $\vec{v} = (7i + 3j)$.

Solution

1) $\vec{u} + \vec{v} = (3i + 4j) + (-2i + j) = (3 - 2)i + (4 + 1)j = 1i + 5j = i + 5j$

2) $5(3i + 2j) - (7i + 3j) = (15i + 10j) - (7i + 3j) = (15 - 7)i + (10 - 3)j = 8i + 7j$

Note

Let $\vec{v} = a_1 i + a_2 j + a_3 k$ and $\vec{w} = b_1 i + b_2 j + b_3 k$ be two vectors and

$$\vec{v} = \vec{w} \Leftrightarrow a_1 = b_1 \text{ and } a_2 = b_2 \text{ and } a_3 = b_3$$

Example: Consider the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ in R^3,
where $P = (2,1,5)$, $Q = (3,5,7)$,
$R = (1,-3,-2)$ and $S = (2,1,0)$. Does $\overrightarrow{PQ} = \overrightarrow{RS}$?

Solution

$\overrightarrow{PQ} = Q - P = (3-2, 5-1, 7-5) = (1,4,2)$.

$\overrightarrow{RS} = (2-1, 1-(-3), 0-(-2)) = (1,4,2)$.

$\therefore \overrightarrow{PQ} = \overrightarrow{RS} = (1,4,2)$.

Example:

Door $\vec{u} = (-2, 1)$, $\vec{v} = (-2, 1)$

1) for $\vec{u} = (5, 3)$, $\vec{v} = (3, 5)$

Solution

1) $\vec{u} = \vec{v}$

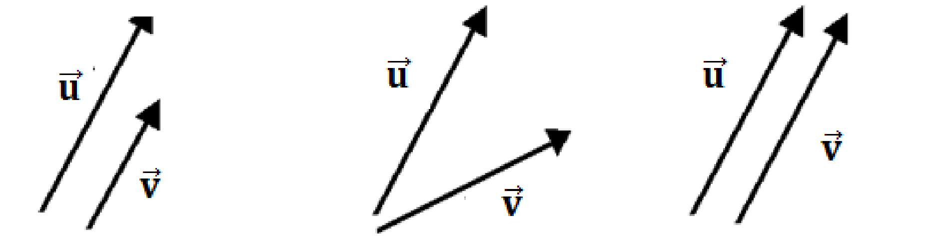
because $u_1 = -2$, $v_1 = -2 \Longrightarrow u_1 = v_1$ **and** $u_2 = 1$, $v_2 = 1 \Longrightarrow u_2 = v_2$

2) $\vec{u} \neq \vec{v}$

because. $u_1 = 5$ $v_1 = 3$, $\Longrightarrow u_1 \neq v_1$, $u_2 = 3$, $v_2 = 5$, $\Longrightarrow u_2 \neq v_2$

Note

Two nonzero vectors are equal if they have the same magnitude and the same direction. Any vector with zero magnitude is equal to the zero vector.



Vector **u** and Vector **v** have same direction but different magnitude.	Vector **u** and Vector **v** have same magnitude but different direction.	Vector **u** and Vector **v** have same direction and same magnitude.
$\vec{u} \neq \vec{v}$	$\vec{u} \neq \vec{v}$	$\vec{u} = \vec{v}$

3-Dot Product

Definition Let $\vec{v} = (v1,v2,v3)$ and $\vec{w} = (w1,w2,w3)$ be vectors in R^3.
The dot product of $\vec{v}$ and $\vec{w}$, denoted by $\vec{v}\cdot\vec{w}$, is given by:

$$\vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2 + v_3w_3$$

Similarly, for vectors $v = (v_1,v_2)$ and $w = (w_1,w_2)$ in R^2, thot product is:

$$\vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2$$

For vectors $v = v_1\, i + v_2\, j + v_3 k$ and $w = w_1\, i + w_2\, j + w_3 k$ in component form, the dot product is still

$$\vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2 + v_3w_3$$

Properties Of The Dot Product

a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (commutative)
b) $(s\vec{u})\cdot\vec{v} = s(\vec{u} \cdot \vec{v})$ (respects scalar multiples)
c) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (distributes over vector sums)
d) $\vec{0} \cdot \vec{u} = 0$
e) $\vec{u} \cdot \vec{v} = 0 \Leftrightarrow \vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$ or $\vec{u} \perp \vec{v}$
f) $s\vec{u} \cdot k\vec{v} = s.k(\vec{u} \cdot \vec{v})$

Note

The associative law does *not* hold for the dot product of vectors Because for vectors u, v, w, the dot product u·v is a scalar, and so (u·v) ·w is not defined since the left side of that dot product (the part in parentheses) is a scalar and not a vector.

Example

given $\vec{u} = (2, -2)$, $\vec{v} = (5, 8)$, $\vec{w} = (-4, 3)$ find each of the following:

1) $\vec{u} \cdot \vec{v}$
2) $(\vec{u} \cdot \vec{v}) \cdot \vec{w}$
3) $\vec{u} \cdot (2\vec{v})$

Solution:

1) $\vec{u} \cdot \vec{v} = (2, -2) \cdot (5, 8) = 2x\,5 + (-2)x8 = 10-16 = -6$
2) $(\vec{u} \cdot \vec{v}) \cdot \vec{w} = -6\,(-4, 3) = (-6\ x\ -4,\ -6\ x\ 3)$
3) $\vec{u} \cdot (2\vec{v}) = 2\,(\vec{u} \cdot \vec{v}) = 2x\ -6 = -12$

The Dot Product Of i, j And k

$i \cdot i = 1 \quad , j \cdot j = 1 \quad , k \cdot k = 1$

$i \cdot j = 0 \quad , j \cdot i = 0 \quad , k \cdot i = 0$

$i \cdot k = 0 \quad , j \cdot k = 0 \quad , k \cdot j = 0$

Define Projection **let be $\vec{A}$ and $\vec{B}$ are vectors the projection of $\vec{B}$ onto $\vec{A}$ $proj_A \vec{B}$**

Is given by

$$proj_A \vec{B} = \frac{\vec{A}.\vec{B}}{|\vec{A}|^2} \vec{A}$$

Note

The projection of $\vec{A}$ onto $\vec{B}$ $proj_A \vec{B}$ is given by

$$proj_B \vec{A} = \frac{\vec{A}.\vec{B}}{|\vec{B}|^2} \vec{B}$$

Example:

Determine the projection of vector $\vec{B}$=(2,1,-1)onto vector $\vec{A}$ =(1,0,-2)

Solution

$$proj_A \vec{B} = \frac{\vec{A}.\vec{B}}{|\vec{A}|^2} \vec{A}$$

$$\vec{A}.\vec{B} = 4$$

$$|\vec{A}| = \sqrt{(1)^2 + (0)^2 + (-2)^2} = \sqrt{5}$$

$$proj_A \vec{B} = \frac{4}{5}(i - 2k)$$

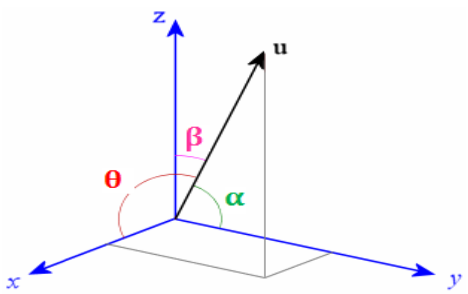
(Direction cosines)

This application direction of the dot product requires that we be in three dimensional spaces unlike all the other application we have looked at to this point

Defection let $\vec{v}$ =(v_1,v_2 , v_3)be a vector three dimensional space and θ is the angle a vector makes with the x axis, α is the angle a vector makes with the y axis, and β is the angle a vector makes with the z axis. These angles are called **Direction angle** and the cosines of these angle are called **Direction cosines**

The formulas for the direction cosines are

$$\boxed{\begin{aligned} \cos\theta &= \frac{v_1}{\|v\|} \Rightarrow v_1=\|v\| \cos\theta \\ \cos\alpha &= \frac{v_2}{\|v\|} \Rightarrow v_2=\|v\| \cos\alpha \\ \cos\beta &= \frac{v_3}{\|v\|} \Rightarrow v_3=\|v\| \cos\beta \end{aligned}}$$



Note

For any vector $\vec{v}$ in Cartesian three-space, the sum of the squares of the direction cosines is always equal to 1.

$$\cos^2\theta + \cos^2\alpha + \cos^2\beta = 1$$

Example

If $\|v\| = 5$ and $\theta=70^\circ$, $\alpha=85^\circ$, $\beta = 20^\circ$ give the component form vector $\vec{v}$.

Solution

$\vec{v}$ = (||v|| cos θ , ||v|| cos α, ||v|| cos β)

= (5 cos70 , 5 cos85, 5 cos20)

Example

Determine the direction cosines and direction angle for $\vec{v} = (2, 1, -4)$

Solution

$$\|\vec{v}\| = \sqrt{4+1+16} = \sqrt{21}$$

$$\cos\theta = \frac{2}{\sqrt{21}} \Rightarrow \theta = 64.123$$

$$\cos\alpha = \frac{1}{\sqrt{21}} \Rightarrow \alpha = 77.396$$

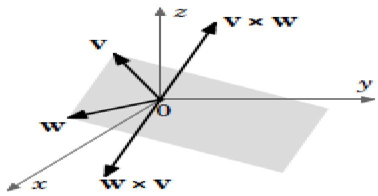
$$\cos\beta = \frac{-4}{\sqrt{21}} \Rightarrow \beta = 150.794$$

4-Cross Product:

Definition let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be vectors in space , the cross product of $\vec{u}$ and $\vec{v}$ is the vector :

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2 \ ,\ u_3v_1 - u_1v_3 \ ,\ u_1v_2 - u_2v_1)$$

The cross product $\vec{u}\times\vec{v}$ of two nonzero vectors $\vec{u}$ and $\vec{v}$ is also a nonzero vector, it is perpendicular to both $\vec{u}$ and $\vec{v}$.



We can now rewrite the definition for the cross product using these determinants:

a) **The top row consists of the unit vectors in order $\vec{i}, \vec{j}, \vec{k}$.**
b) **The second row consists of the coefficients $\vec{u}$.**

c) **The third row consists of the coefficients $\vec{v}$.**

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \begin{array}{l} \leftarrow \text{Put "}\mathbf{u}\text{" in Row 2.} \\ \leftarrow \text{Put "}\mathbf{v}\text{" in Row 3.} \end{array}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\mathbf{k}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Properties of the cross product:

If $\vec{v}$, $\vec{u}$, and $\vec{w}$ are vectors and s is a scalar, then :

1. $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$ **(Anti-commutative)**

2. $(s\,\vec{v}) \times \vec{u} = s(\vec{v} \times \vec{u}) = \vec{v} \times (s\vec{u})$

3. $\vec{v} \times (\vec{u} + \vec{w}) = \vec{v} \times \vec{u} + \vec{v} \times \vec{w}$ **(Distributive)**

4. $(\vec{v} + \vec{u}) \times \vec{w} = \vec{v} \times \vec{w} + \vec{u} \times \vec{w}$

5. $\vec{v} \cdot (\vec{u} \times \vec{w}) = (\vec{v} \times \vec{u}) \cdot \vec{w} = \vec{u} . (\vec{w} \times \vec{v})$

6. $\vec{v} \times (\vec{u} \times \vec{w}) \neq (\vec{v} \times (\vec{u}) \times \vec{w}$ **(Not associative)**

7. $\vec{v} \times (\vec{u} \times \vec{w}) = (\vec{v} \cdot \vec{w})\,\vec{u} - (\vec{v} \cdot \vec{u})\,\vec{w}$ **(both sides of this identity are vectors)**

Note

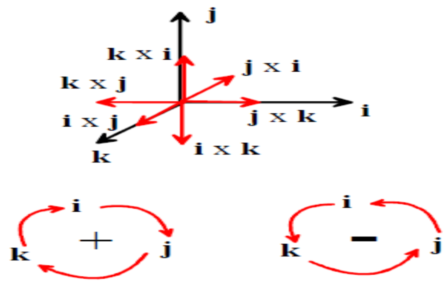
For any vectors $\vec{u} = (u_1,u_2,u_3)$, $\vec{v} = (v_1,v_2,v_3)$, $\vec{w} = (w_1,w_2,w_3)$ in $\mathbf{R}^3$:

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_2\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

The Cross Product Of i, j And k

$\mathbf{i}\times\mathbf{j} = \mathbf{k}$, $\mathbf{j}\times\mathbf{k} = \mathbf{i}$, $\mathbf{k}\times\mathbf{i} = \mathbf{j}$
$\mathbf{i}\times\mathbf{i} = 0$, $\mathbf{j}\times\mathbf{j} = 0$, $\mathbf{k}\times\mathbf{k} = 0$
$\mathbf{i}\times\mathbf{k} = -\mathbf{j}$, $\mathbf{j}\times\mathbf{i} = -\mathbf{k}$, $\mathbf{k}\times\mathbf{j} = -\mathbf{i}$

to find the cross product of any pair of basis vectors ,you travel around the circle. Thus, to get i x j , you start at i, move to j and then on to k. If you go around the circle clockwise, the answer is positive, if you go counter-clockwise, it is negative. Thus, j x k =i , and so on, while k x j =-i , etc.



Example:

Given $\vec{u}$ = i – 2j + k and $\vec{v}$ = 3i + j – 2k, find each of the following.

a. $\vec{u}$ x v   b. $\vec{v}$ x $\vec{u}$ c. $\vec{v}$ x $\vec{v}$

Solution

a. $\vec{u}$ x v $= \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix}$

$= \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} i - \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} j + \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} k$

$= (4 - 1)\, i - (-2-3)j + (1+6)k = 3i + 5j + 7k$

b. $\vec{v}$ x $\vec{u}$ $= \begin{vmatrix} i & j & k \\ 3 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} i - \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} j + \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} k$

$= (1 - 4)\, i - (3+2)j + (-6-1)k = -3i - 5j - 7k$

c. $\vec{v}$ x $\vec{v}$ $= \begin{vmatrix} i & j & k \\ 3 & 1 & -2 \\ 3 & 1 & -2 \end{vmatrix}$

$= \begin{vmatrix} 1 & -2 \\ 1 & -2 \end{vmatrix} i - \begin{vmatrix} 3 & -2 \\ 3 & -2 \end{vmatrix} j + \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} k$

$= (-2 + 2)\, i - (-6+6)j + (3-3)k = 0i - 0j - 0k = 0$

Example:

Given the vectors $\vec{v} = i - 2j + 4k$ and $\vec{w} = 3i + j - 2k$ find $\vec{v} \times \vec{w}$

Solution

$$\vec{v}\times\vec{w} = \begin{vmatrix} i & j & k \\ 1 & -2 & 4 \\ 3 & 1 & -2 \end{vmatrix}$$

$$= i(4-4) - j(-2-12) + k(1+6) = 14j + 7k$$

$$\vec{w}\times\vec{v} = \begin{vmatrix} i & j & k \\ 3 & 1 & -2 \\ 1 & -2 & 4 \end{vmatrix} = = i(4-4) - j(12+2) + k(-6-1) = -14j - 7k$$

Example:

Given the vectors $\vec{v} = j + 6k$ and $\vec{w} = i + j$ find $\vec{v}\times\vec{w}$

Solution

$$\vec{v}\times\vec{w} = \begin{vmatrix} i & j & k \\ 0 & 1 & 6 \\ 1 & 1 & 0 \end{vmatrix} = i(1.0-6) - j(0-6) + k(0-1) = -6i+6j-k$$

Example:

Find $\vec{u}\times(\vec{v}\times\vec{w})$ for $\vec{u} = (1,2,4)$, $\vec{v} = (2,2,0)$, $\vec{w} = (1,3,0)$.

Solution:

Since $\vec{u}\cdot\vec{v} = 6$ and $\vec{u}\cdot\vec{w} = 7$, then

$$\vec{u}\times(\vec{v}\times\vec{w}) = (\vec{u}\cdot\vec{w})\,\vec{v} - (\vec{u}\cdot\vec{v})\,\vec{w}$$

$$= 7(2,2,0) - 6(1,3,0) = (14,14,0) - (6,18,0)$$

$$= (8,-4,0)$$

Note

Angle between Vectors

Let $\vec{u}$ and $\vec{v}$ be from R^2 or R^3 and let θ be the angle between them. Then

- **The angle between two vectors can be found by using the dot product.**

$$\cos(\theta) = \frac{\vec{u}.\vec{v}}{|u|.|v|} \Longrightarrow \theta = \cos^{-1}\left(\frac{\vec{u}.\vec{v}}{|u|.|v|}\right)$$

- **The angle between two vectors can be found by using the cross product.**

$$\sin\theta = \frac{|\vec{v}\times\vec{u}|}{|u||v|} \Longrightarrow \theta = \sin^{-1}\left(\frac{|\vec{v}\times\vec{u}|}{|v||u|}\right)$$

$$\vec{u}\cdot\vec{v} \text{ is } \begin{cases} positive & o \le \theta < \frac{\pi}{2} \\ 0 & \theta = \frac{\pi}{2} \\ negative & \frac{\pi}{2} < \theta \le \pi \end{cases}$$

Example

What is the angle in degrees between $\vec{u}$ =(1, 1, 1) and $\vec{v}$ = (2 ,1 , 0),

Solution

$\vec{u}\cdot\vec{v} = 1.2 + 1.1 + 1.0 = 3$

$$|u| = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$|v| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{2^2 + 1^2 + 0^2} = \sqrt{5}$$

$$\cos(\theta) = \frac{\vec{u}.\vec{v}}{|u|.|v|} = \frac{3}{\sqrt{3}\sqrt{5}} = \frac{3}{\sqrt{15}} \Rightarrow \theta = \cos^{-1}\left(\frac{3}{\sqrt{15}}\right) = 39{:}23^{o}$$

Example:

Find the angle between $\vec{u}$. = 2i+3j+k & $\vec{v}$= -i+5j+k.

Solution:

$\vec{u}.\ \vec{v}$=2.(-1)+3.5+1.1 =-2+15+1 =14

$\|\vec{u}\| = \sqrt{14}$, $\|\vec{v}\| = \sqrt{27}$

$$\cos(\theta) = \frac{\vec{u}.\vec{v}}{|u|.|v|} = \frac{14}{\sqrt{14}.\sqrt{27}} = \frac{\sqrt{14}}{3\sqrt{3}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{\sqrt{14}}{3\sqrt{3}}\right) = 43.93^{o} \simeq 44^{o}$$

Example

Given $\vec{u}$ = 2i − 3j + 5k and $\vec{v}$ = 5i + 3j − 7k, compute the angle between $\vec{u}$ and $\vec{v}$

Solution

$$\cos\theta = \frac{\vec{u}.\vec{v}}{|u|.|v|}$$

$\vec{u} \,.\, \vec{v}$ = 2 x 5 + (-3)x 3 +5x(-7)= -34

$$|u| = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{2^2 + -3^2 + 5^2} = \sqrt{38}$$

$$|v| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{5^2 + 3^2 + -7^2} = \sqrt{83}$$

$$\cos(\theta) = \frac{\vec{u}.\vec{v}}{|u|.|v|} = \frac{-34}{\sqrt{38}\sqrt{83}} = \frac{-34}{\sqrt{3154}}$$

Note

One of the application of cross product to find unit vector normal $(\vec{N})$on $\vec{A}$ and $\vec{B}$

$$\vec{N} = \frac{\vec{A}x\vec{B}}{|\vec{A}x\vec{B}\,|}$$

Example:

Find the normal unit vector perpendicular on $\vec{A}$ and $\vec{B}$ for

$\vec{A} = 2i + 3j - k$ **and** $\vec{B} = -j + 2k$

Solution

$$\vec{N} = \frac{\vec{A}x\vec{B}}{|\vec{A}x\vec{B}\,|}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 5i - 4j - 2k$$

$$|\vec{A} \times \vec{B}| = \sqrt{(5)^2 + (-4)^2 + (-2)^2} = \sqrt{45}$$

$$\vec{n} = \frac{5}{\sqrt{45}}i - \frac{4}{\sqrt{45}}j - \frac{2}{\sqrt{45}}k$$

Parallel Vectors

- **Two nonzero vectors $\vec{v}$ and $\vec{u}$ are parallel if there is some scalar c such that**

$$\vec{u} = c\vec{v}$$

Or

- **Two non-zero vectors $\vec{v}$ and $\vec{u}$ are parallel $\Leftrightarrow$ $\vec{v}\times\vec{u} = 0$.**

Example

Which of the following vectors is parallel to $\vec{w}$ = (– 6 , 8, 2)?
a. $\vec{u}$ = (3, –4, –1)
b. $\vec{v}$ =(12, –16, 4)

Solution

a. Because $\vec{u}$ = (3 ,– 4 , – 1) = $\frac{1}{2}$(– 6, 8, 2) = $-\frac{1}{2}\vec{w}$, you can conclude that u is parallel to w.

OR

$$\vec{w}x\vec{u} = \begin{vmatrix} i & j & k \\ -6 & 8 & 2 \\ 3 & -4 & -1 \end{vmatrix} = (\,8x\text{-}1 - 2x\text{-}4, 2x3 - \text{-}6x\text{-}1, \text{-}6x\text{-}4\text{-}8x3) = (0\,,\ 0,\ 0)$$

b. In this case, you want to find a scalar *c* such that

(12, –16, 4) = c(–6, 8, 2) .
12 = –6→ *c* = –2
–16 = 8→ *c* = –2
4 = 2→ *c* = 2
Because there is no *c* for which the equation has a solution ,the vectors are not parallel.

Orthogonal Vector

- **Two non-zero vectors $\vec{v}$ and $\vec{u}$ are orthogonal $\Leftrightarrow$ $\vec{u} \cdot \vec{v} = 0$**

Example

Show that vectors $\vec{v}$=(1, -1, 0) and $\vec{w}$ = (2,2 ,4) are orthogonal,

Solution

$$\vec{v}.\vec{w} = 1*2+-1*2+0*4=0$$

Example

Determine whether the given vectors are orthogonal, parallel or neither:

(1) $\vec{u}$ **=(-2, 6, -4) ,**$\vec{v}$**= (4 ,-12 , 8).**

(2) $\vec{u}$**=i-j+2k , $\vec{v}$ =2i-j+k.**

(3) $\vec{u}$**=(a ,b, c) , $\vec{v}$ =(-b, a , 0)**

Solution

(1) $\vec{u}$ **=(-2 ,6, -4) , $\vec{v}$ = (4, -12, 8).**

Because $\vec{v}$ = (4, -12 , 8) = - 2 (-2, 6 ,-4)= 2 $\vec{u}$, you can conclude that $\vec{v}$ is parallel to $\vec{u}$

(2) conclude that $\vec{u}$ is parallel to $\vec{v}$

$\vec{u}.\vec{v}$ = 2+1+2 = 5 ,

$$\vec{u}x\vec{v}=\begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 2 & -1 & 1 \end{vmatrix}= (-1x1 – 2x-1 ,\ 1x1 - 2x2 ,\ 1x-1- -1x2) =(1 , 3, 1)$$

the vectors are neither orthogonal nor parallel.

(3) $\vec{u}$**=(a, b, c) , $\vec{v}$ =(-b ,a , 0).**

$\vec{u}$. $\vec{v}$ = -ab + ab + 0 = 0, so the vectors are orthogonal.

Parametric Equations

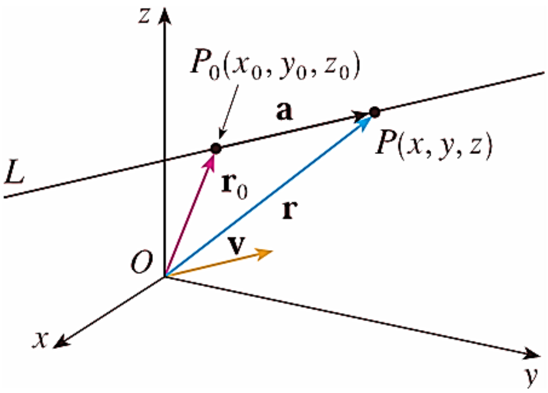
The parametric equations of a lineL in 3-space for a line passing through p_0 (x_0,y_0,z_0) and parallel to v = ai + bj + ck:

is $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$

The above equation are called parametric equations for the line

To determine parametric equations of a line, we need

* a point on the line
* a vector parallel to the line



Vector Equation

The vector equation of the line is: $\vec{r} = \vec{r_0} + t\,\vec{v}$

Where $\vec{r_0}$ = (x_0, y_0, z_0) is a vector whose components are made of the point (x_0, y_0, z_0) on the line L and

$\vec{v}$ = (a, b, c) are components of a vector that is parallel to the line L

Example:

Find the parametric equation of the line passing through $p_0(1,3,2)$ parallel to 2i−j+3k.

Solution:

The parametric equation is

$$x = 1 + 2t$$
$$y = 3 - t$$
$$z = 2 + 3t$$

Example:

Find a vector equation and parametric equations for the line that passes through the point (5, 1, 3) and is parallel to the vector $\vec{v}$ = i + 4j − 2k.

Solution

Here $\overrightarrow{r_0}$ = (5, 1, 3) = 5i + j + 3k and $\vec{v}$= i + 4j − 2k, so the vector

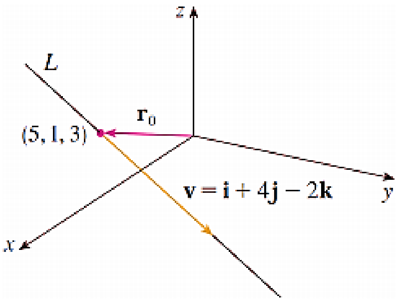
$$\vec{r} = \overrightarrow{r_0} + t\vec{v}$$

becomes

$$\vec{r} = (5i + j + 3k) + t(i + 4j - 2k)$$
$$\vec{r} = (5 + t)i + (1 + 4t)j + (3 - 2t)k$$

Parametric equations are

$$x = 5 + t \quad , \quad y = 1 + 4t \quad , z = 3 - 2t$$



symmetric the equations

Consider the parametric form equations for a line:

L : $x = x_0 + at$, $y = y_o + bt$, $z = z_0 + ct$.

If a, b and c are all nonzero, we can solve each equation for t to get :

$\frac{x-x_0}{a} = t$

$\frac{y-y_0}{b} = t$

$\frac{z-z_0}{c} = t$

We called these three equation symmetric form of the system of equations for line L, If we set:

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} = t$$

If one or more of a, b and c is zero, we can still obtain symmetric equations. For example, if a = 0, the symmetric equations are

$$x = x_0 \, , \frac{y-y_0}{b} = \frac{z-z_0}{c} = t$$

Example:

Find the symmetric equation for line through point (1,-5,6) and is parallel to vector (-1,2,-3)

Solution

$$\frac{x-1}{-1} = \frac{y+5}{2} = \frac{z-6}{-3}$$

Equations of Planes

The equation of the plane in 3 space, that passing through the point $P_0(x_0, y_0, z_0)$ on the Plane and the non zero vector $\vec{n} = ai + bj + ck$ Perpendicular (orthogonal) to the plane (The vector $\vec{n}$ is called normal Vector) is $ax + by + cz = D; where\ D = ax_0 + by_0 + cz_0$

Note

- To find the equation of a plane in R^3, we need to know:
  1. A point on the plane $P_0(x_0, y_0, z_0)$.
  2. A normal (perpendicular) vector to the plane.

Example:

Find the equation of plane through point (1,-1,1) and with normal vector i+j-k

Solution

Given point is (1,-1,1)

Here a=1,b=1,c=-1

We know that equation of plane is given by-:

$$ax + by + cz = D; where\ D = ax_0 + by_0 + cz_0$$

$$x + y - z = 1$$

Example: .

Find the equation of the plane with normal $\vec{n}$ = (1, 2, 7) which contains the point P_0 (5, 3, 4)

Solution

The equation of plane is

x+2y+7z=39

Example:

Determine the equation of the plane that contains the points

P_1 = (1, -2 , 0) , P_2 = (3 , 1, 4) , P_3 = (0, -1, 2)

Solution

In order to write the equation of plane we need a point and a normal vector , We need to find a normal vector.

Step 1

First convert the three points into two vectors by subtracting one point from the other two

$\overrightarrow{P_1P_2}$ **= (3-1, 1-(-2) , 4-0) = (2, 3, 4)**

$\overrightarrow{P_1P_3}$ **= (0-1, -1 –(-2) , 2-0) = (-1 , 1, 2)**

Step 2

Find the cross product of the vectors found in Step 1. we know that the cross product of two vectors will be orthogonal to both of these vectors. Since both of these are in the plane any vector that is orthogonal to both of these will also be orthogonal to the plane. Therefore, we can use the cross product as the normal vector.

$$\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ -1 & 1 & 2 \end{vmatrix} = 2\,i - 8j + 5k$$

Step 3

The coefficients a, b, and c of the planar equation are (2 -8 5), then We can used any of the three points to find The equation of the plane >

$2x - 2 - 8y - 16 + 5z = 0$

$2x - 8y + 5z = 18$

Example:

Find an equation of the plane which contains the points $P_1\ (-1, 0, 1)$, $P_2\ (1, -2, 1)$ and $P_3\ (2, 0, -1)$.

Solution

Step 1

$\overrightarrow{P_1P_2} = (2, -2, 0)$

$\overrightarrow{P_1P_3} = (3, 0, -2)$

Step 2

$$\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} i & j & k \\ 2 & -2 & 0 \\ 3 & 0 & -2 \end{vmatrix} = 4i + 4j + 6k$$

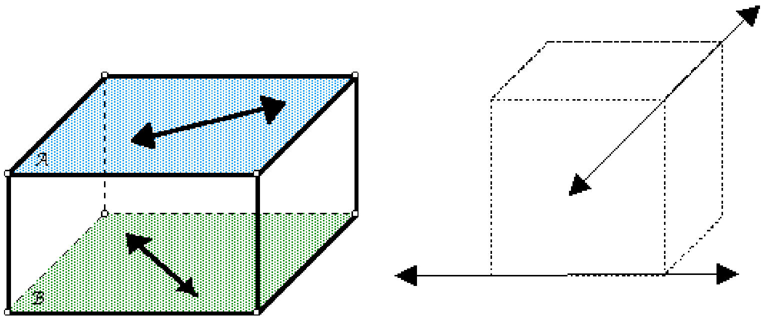
Step 3

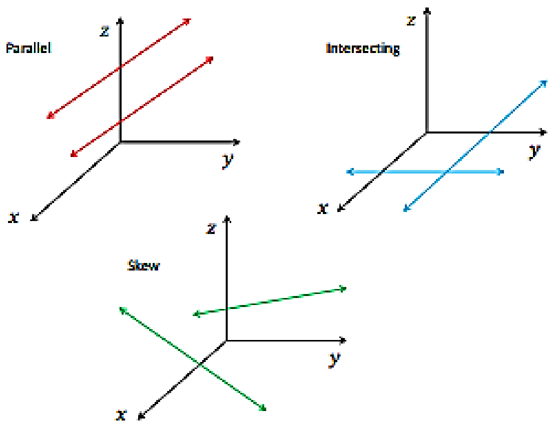
$\vec{n} = (4, 4, 6)$

$4x + 4 + 4y + 6z - 6 = 0$

$4x + 4y + 6z = 2$

In two dimensions, two lines either intersect or are parallel; in three dimensions, lines that do not intersect might not be parallel.
Two lines that are not parallel and do not intersect are called skew lines.





Example:

Show that the lines

L_1: $x = t_1 - 1,\ y = t_1 + 5,\ z = 1$

L_2: $x = t_2 - 3,\ y = -t_2 + 1,\ z = t_2 + 2$

Intersect , and find the point of intersection .

Solution:

If they intersect, we can find a value of t_1 and t_2 that satisfy the equations

$x_0 = t_1 - 1 = t_2 - 3$ -------------------------(1)

$y_0 = t_1 + 5 = -t_2 + 1$----------------------(2)

$z_0 = 1 = t_2 + 2$----------------------------(3)

from equation (3)

$1 = t_2 + 2 \Longrightarrow t_2 = -1$

From equation (1) or (2)

$t_1 - 1 = t_2 - 3$

$t_1 - 1 = -1 - 3 \Longrightarrow t_1 = -4+1 = -3$

Then check whether the three sets of equations are satisfied by (t_2, t_1)

$x_0 = t_1 - 1 = t_2 - 3 \quad \Longrightarrow -3-1 = -1-3 \qquad \Longrightarrow -4=-4$

$y_0 = t_1 + 5 = -t_2 + 1 \Longrightarrow -3+5 = -(-1)+1 \Longrightarrow 2 = 2$

$z_0 = 1 = t_2 + 2 \quad \Rightarrow 1 = -1+2 \quad \Rightarrow 1 = 1$

The point of intersection $(x_0, y_0, z_0) = (-4, 2, 1)$

Example:

Let L_1 and L_2 be lines with parametric equations

L_1 : $x = 1 + 2t_1$; $y = 3 + 2t_1$; $z = 2 - t_1$

L_2 : $x = 2 + t_2$; $y = 6 - t_2$; $z = -2 + 3t_2$

Determine whether the lines are parallel, skew, or intersecting. If they intersect, find the point of intersection

Solution:

The direction vectors are $\vec{v}_1 = (2; 2;-1)$ and $\vec{v}_2 = (1;-1; 3)$

$\vec{v}_1 \neq c\vec{v}_2$

So these vectors are not parallel . Do they intersect

If there is an intersection point $(x_0; y_0; z_0)$, we will find it by solving the system of three equations in parameters t_1 and t_2:

$x_0 = 1 + 2t_1 = 2 + t_2$ -------------------- (1)

$y_0 = 3 + 2t_1 = 6 - t_2$ -------------------- (2)

$z_0 = 2 - t_1 = -2 + 3t_2$ ------------------- (3)

solve the first two equation for t_1 , t_2 .

$1 + 2t_1 = 2 + t_2$ --------------------- (1)

$3 + 2t_1 = 6 - t_2$ --------------------- (2)

Subtract equation (2) from equation (1) we get

$-2 = -4+2t_2 \Rightarrow 2t_2 = -2+4$

$t_2 = 1$

We can find t_1 by substituting this value of t_2 in either the first or second equation

$1 + 2t_1 = 2 + t_2 \Rightarrow 1 + 2t_1 = 2 + 1$

$2t_1 = 2 \Rightarrow t_1 = 1$

Then check whether the three sets of equations are satisfied by (t_2, t_1)

$x_0 = 1 + 2t_1 = 2 + t_2 \Rightarrow 1+ 2= 2+1 \quad \Rightarrow 3=3$

$y_0 = 3 + 2t_1 = 6 - t_2 \Rightarrow 3+2= 6-1 \Rightarrow 5 = 5$

$z_0 = 2 - t_1 = -2 + 3t_2 \Rightarrow 2-1= -2+3 \quad \Rightarrow 1 = 1$

The point of intersection $(x_0, y_0, z_0) = (3, 5, 1)$

Example:

Determine whether the lines

L_1 : $x = 1 + 2t_1$; $y = 3t_1$; $z = 2- t_1$

L_2 : $x = -1 + t_2$; $y = 4 + t_2$; $z = 1 + 3t_2$

parallel, skew or intersecting.

Solution

The direction vectors are $\vec{v}_1 = (2, 3,-1)$ and $\vec{v}_2 = (1, 1, 3)$

$\vec{v}_1 \neq c\vec{v}_2$

So these vectors are not parallel . Do they intersect

If there is an intersection point (x_0; y_0; z_0), we will find it by solving the system of three equations in parameters t_1 and t_2

$x_0 = 1 + 2t_1 \quad = -1 + t_2$ ----------------------- (1)

$y_0 = 3t_1 = 4 + t_2$ ----------------------- (2)

$z_0 = 2 - t_1 = 1 + 3t_2$ ----------------------- (3)

Let us solve the first two equations.

$1 + 2t_1 \; = -1 + t_2$ ----------------------- (1)

$3t_1 \quad = 4 + t_2$ ----------------------- (2)

subtract equation (2) from equation (1) we get

$1 - t_1 = -5 \Longrightarrow \quad t_1 = 6$

we can find t_2 by substituting this value of t_1 in either the first or second equation

$3t_1 \quad = 4 + t_2$

$3(6) \quad = 4 + t_2 \Longrightarrow \; t_2 = 18 - 4 = 14$

Then check whether the three sets of equations are satisfied by (t_2, t_1)

$x_0 = 1 + 2t_1 \quad = -1 + t_2 \Longrightarrow 1 + 2(6) \quad = -1 + 14 \Longrightarrow 13 = 13$

$y_0 = 3t_1 = 4 + t_2 \quad \Longrightarrow \quad 3(6) = 4 + 14 \quad \Longrightarrow \quad 18 = 18$

$z_0 = 2 - t_1 = 1 + 3t_2 \quad \Longrightarrow \quad 2 - 6 = 1 + 3(14) \quad \Longrightarrow -4 \; = 43$

The solution does not satisfy the third equation. So these lines do not intersect, therefore, they are sk

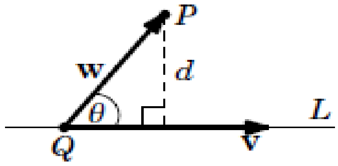
Distance between a Point and a Line

Let *L* be a line and let *P* be a point not on *L*. The distance *d* from *P* to *L* is the length of the line segment from *P* to *L* which is perpendicular to *L* ,Pick a point P_0 on *L*, and let w be the vector from P_0 to *P*. If θ is the angle between w and v, then

d = |*w* |sin θ.

$\because |\vec{v} \times \vec{w}| = |\vec{v}|\,|\vec{w}| \sin\theta$

$\therefore |\vec{w}| \sin\theta = \dfrac{|\vec{v}\times\vec{w}|}{|\vec{v}|} \implies d = \dfrac{|\vec{v}\times\vec{w}|}{|\vec{v}|}$



Example:

Find the distance *d* from the point *P* = (1,1,1) to the line

$L : x = -3+7t,\ y = 1+3t,\ z = -4-2t$

Solution

The distance *d* from *P* to *L* :

$d = \dfrac{|\vec{v}\times\vec{w}|}{|\vec{v}|}$

$\vec{v} = (7, 3, -2)$

$\vec{w} = \overrightarrow{P_0P} = (1-(-3),\ 1-1,\ 1-(-4)) = (4, 0, 5)$

$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 7 & 3 & -2 \\ 4 & 0 & 5 \end{vmatrix} = 15i - 43j - 12k$

$|\vec{v} \times \vec{w}| = \sqrt{15^2 + -43^2 + -12^2} = \sqrt{2218}$

$|\vec{v}| = \sqrt{7^2 + 3^2 + -2^2} = \sqrt{49 + 9 + 4} = \sqrt{62}$

$\therefore d = \dfrac{|\vec{v}\times\vec{w}|}{|\vec{v}|} = \dfrac{\sqrt{2218}}{\sqrt{62}} = 5.98$

Example:

Find the distance *d* from the point *P* = (1, 4, -3) to the line

L : x=2+t, y=-1-t, z=3t

Solution

The distance *d* from *P* to *L* :

$$d = \frac{|\vec{v} \times \vec{w}\,|}{|\vec{v}|}$$

$\vec{v}$ = (1 ,-1, 3)

$\vec{w} = \overrightarrow{P_0P}$ = (1- 2 , 4- (-1), -3- 0) = (-1 ,5, -3)

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 1 & -1 & 3 \\ -1 & 5 & -3 \end{vmatrix} = -12i - 6j + 6k$$

$$|\vec{v} \times \vec{w}| = \sqrt{(-12)^2 + -6^2 + 6^2} = \sqrt{216}$$

$$|\vec{v}| = \sqrt{1^2 + -1^2 + 3^2} = \sqrt{1+1+9} = \sqrt{11}$$

$$\therefore d = \frac{|\vec{v} \times \vec{w}\,|}{|\vec{v}|} = \frac{\sqrt{216}}{\sqrt{11}} = 4.43$$

Distance between two lines

Let P_1 be a point and $\vec{v_1}$ be a direction vector for a line L_1 and

let P_2 be point and $\vec{v_2}$ be a direction vector for a line L_2 .

The distance between two parallel line

If $\vec{v_1} \times \vec{v_2} = 0$ OR $\vec{v_1} = c\,\vec{v_2} \Longrightarrow L_1 \parallel L_2$

$$d = \frac{|\vec{v_1} \times \overrightarrow{P_1P_2}\,|}{|\vec{v_1}\,|}$$

The distance between two intersection line

If $\vec{v_1} \times \vec{v_2} \neq 0$ & $\overrightarrow{P_1P_2}.(\vec{v_1} \times \vec{v_2}) = 0 \Longrightarrow L_1 \cap L_2$

d=0

The distance between two skew line

If $\vec{v_1} \times \vec{v_2} \neq 0$ & $\overrightarrow{P_1P_2}.(\vec{v_1} \times \vec{v_2}) \neq 0 \Longrightarrow$ the two lines are skew

$$d = \frac{|\overrightarrow{P_1P_2}.\overrightarrow{(\vec{v_1} \times \vec{v_2})}\,|}{|\vec{v_1} \times \vec{v_2}|}$$

Example:

Find the distance *d* between the two lines:

1. L_1 : x=1+2t, y=2+t, z=-3+3t

   L_2 : x=2+10t, y=-2+5t, z=3+15t

2. L_1 : x=1+2t, y=2+2t, z=-3+3t
L_2 : x=2+t, y=-2-t, z=3+7t
3. L_1 : x=1+t, y=1-2t, z=8+t
L_2 : x=3t, y=2+5t, z=8-8t

Solution

1. L1 : x=1+2t, y=2+t, z=-3+3t
L2 : x=2+10t, y=-2+5t, z=3+15t
The direction vectors are $\vec{v}_1$ = (2 ,1 , 3) and $\vec{v}_2$ = (10, 5 ,15)
$\vec{v}_1 = 5\vec{v}_2$
So these vectors are parallel

$$d = \frac{|\vec{v_1} \times \overrightarrow{P_1P_2}\,|}{|\vec{v_1}|}$$

$\overrightarrow{P_1P_2}$ = (2-1 , -2-2, 3- (-3))= (1 ,-4, 6)

$$\vec{v_1} \times \overrightarrow{P_1P_2} = \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ 1 & -4 & 6 \end{vmatrix} = 18i - 9j - 9k$$

$$\left|\vec{v_1} \times \overrightarrow{P_1P_2} = \right| = \sqrt{18^2 + -9^2 + -9^2} = 9\sqrt{6}$$

$$|\vec{v_1}| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$$

$$d = \frac{9\sqrt{6}}{\sqrt{14}} = 5.9$$

2. L_1 : x=1+2t, y=2+2t, z=-3+3t
L_2 : x=2+t, y=-2-t, z=3+7t

The direction vectors are $\vec{v}_1$ = (2 , 2, 3) and $\vec{v}_2$ = (1 , -1, 7)
$\vec{v}_1 \neq c\vec{v}_2$
So these vectors are not parallel

$\overrightarrow{P_1P_2}$= (2-1 , -2-2 , 3 – (-3)) = (1 , -4 , 6)

$$\vec{v_1} \text{ x } \vec{v_2} = \begin{vmatrix} i & j & k \\ 2 & 2 & 3 \\ 1 & -1 & 7 \end{vmatrix} = 17i - 11j - 4k$$

$\overrightarrow{P_1P_2}.(\vec{v_1} \text{ x } \vec{v_2})$ = 1 x 17 +(-4 x 11) +6 x (-4) = 17-44-24 =37 $\neq$ 0

$\vec{v_1} \text{ x } \vec{v_2} \neq 0$ & $\overrightarrow{P_1P_2}.(\vec{v_1} \text{ x } \vec{v_2}) \neq 0 \Longrightarrow$ the two lines are skew

$$d = \frac{|\overrightarrow{P_1P_2}.(\vec{v_1} \text{ x } \vec{v_2})\,|}{|\vec{v_1} \text{ x } \vec{v_2}|}$$

$|\vec{v_1} \times \vec{v_2}| = \sqrt{17^2 + -11^2 + -4^2} = \sqrt{426}$

$\therefore d = \frac{|37|}{\sqrt{426}} = \frac{37}{\sqrt{426}} = 1.79$

3. $L_1 : x=1+t, \quad y=1-2t, \quad z=8+t$

$L_2 : x=3t, \quad y=2+5t, \quad z=8-8t$

The direction vectors are $\vec{v}_1 = (1, -2, 1)$ and $\vec{v}_2 = (3, 5, -8)$

$\vec{v}_1 \neq c\vec{v}_2$

So these vectors are not parallel

$\overrightarrow{P_1P_2} = (0-1, 2-1, 8-8) = (-1, 1, 0)$

$\vec{v_1} \times \vec{v_2} = \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 3 & 5 & -8 \end{vmatrix} = 11i + 11j + 11k$

$\overrightarrow{P_1P_2}.(\vec{v_1} \times \vec{v_2}) = -1 \times 11 + 1 \times 11 + 0 \times 11 = 0$

$\vec{v_1} \times \vec{v_2} \neq 0 \ \& \ \overrightarrow{P_1P_2}.(\vec{v_1} \times \vec{v_2}) = 0 \Rightarrow L_1 \cap L_2$

$\therefore d = 0$

Distance between Point and Plane

The distance D between a point $p_0 = (x_0, y_0, z_0)$ and the plane; $ax + by + cz + d = 0$ is

$$D = \frac{|a\,x_1 + b\,y_1 + c\,Z_1 + d|}{\sqrt{a^2+b^2+c^2}}$$

Example:

Find the distance D from point (2,4,−5) to the plane

$5x - 3y + z - 10 = 0$

Solution

$$D = \frac{|a\,x_1 + b\,y_1 + c\,Z_1 + d|}{\sqrt{a^2+b^2+c^2}} =$$

$$D = \frac{|5x - 3y + z - 10|}{\sqrt{5^2 + -3^2 + 1^2}} = \frac{|5(2) - 3(4) + (-5) - 10|}{\sqrt{35}} = \frac{|-17|}{\sqrt{35}} = \frac{17}{\sqrt{35}} = 2.87$$

Example:

Find the distance D from point(1, 6, -1)to the plane

2x + y - 2z - 19= 0

Solution

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$$D = \frac{|a\,x_1 + b\,y_1 + c\,Z_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$D = \frac{|2x + y - 2z - 19|}{\sqrt{2^2 + 1^2 + -2^2}} = \frac{|2(1) + 6 - 2(-1) - 19|}{\sqrt{9}} = \frac{|-9|}{\sqrt{9}} = \frac{9}{3} = 3$$