وزارة التعليم العالي والبحث العلمي
الجامعة التقنية الوسطى
الكلية التقنية الهندسية الكهربائية
قسم هندسة تقنيات الحاسوب

Control Engineering Fundamentals

Lesson1

Fundamentals of Control Systems

Fundamentals of Control Systems

Objectives:

At the end of this lesson, students will be able to:

1. Define the control system.
2. classify the different types of control systems.

Basics of Control Systems

System : *A system is a combination or an arrangement of different physical components which act together as an entire unit to achieve certain objective.*

Control system : *To control means to regulate, to direct or to command. Hence a control system is an arrangement of different physical elements connected in such a manner so as to regulate, direct or command itself or some other system.*

Fig. 1.1 Physical system

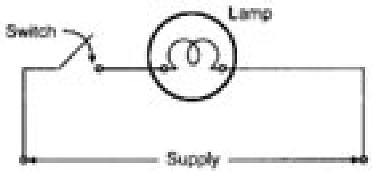


Fig. 1.2 Control system

Definitions:

Plant : *The portion of a system which is to be controlled or regulated is called the* ***plant*** *or the* ***Process.***

Controller : *The element of the system itself or external to the system which controls the plant or the process is called* **controller**.

Input : *It is an applied signal or an excitation signal applied to a control system from an external energy source in order to produce a specified output.*

Output : *It is the particular signal of interest or the actual response obtained from a control system when input is applied to it.*

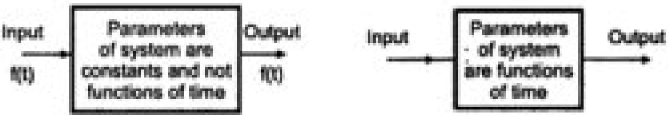
Disturbances : Disturbance is a signal which tends to adversely affect the value of the output of a system. If such a disturbance is generated within the system itself, it is called an **internal disturbance**. The disturbance generated outside the system acting as an extra input to the system in addition to its normal input, affecting the output adversely is called an **external disturbance**.

The input variable is generally referred as the **Reference Input** and output is generally referred as the **Controlled Output**.

Fig. 1.3

Classification of Control Systems

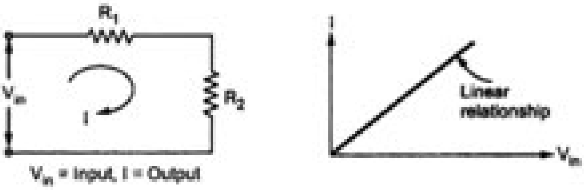
1. Time variant and Time invariant systems



(a) Time invariant system (b) Time variant system

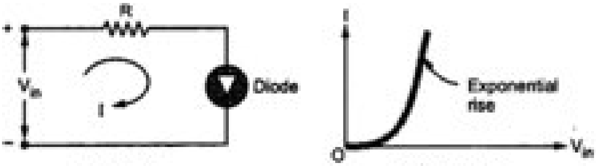
Fig. 1.4

2. Linear and Nonlinear systems



(a) Linear system (b) Response of system

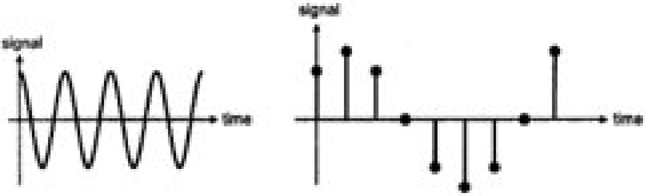
Fig. 1.5 Example of linear system



(a) System (b) Response

Fig. 1.6 Example of nonlinear system

3. Continuous Time and Discrete Time systems



(a) Continuous signal (b) Discrete signal

Fig. 1.7

4. Single Input Single Output (SISO) and Multiple Input Multiple Output (MIMO) systems

5. Open loop and Closed loop systems

5.1 Open Loop System

Definition : A system in which output is dependent on input but controlling action or input is totally independent of the output or changes in output of the system, is called an Open Loop System.

In a broad manner it can be represented as in Fig. 1.9. 8

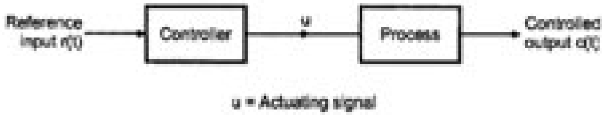


Fig. 1.8 Open loop control system

Reference input [r(t)] is applied to the controller which generates the actuating signal (u) required to control the process which is to be controlled. Process is giving out the necessary desired controlled output c(t).

5.1.1 Advantages

The advantages of open loop control system are,

1) Such systems are simple in construction.
2) Very much convenient when output is difficult to measure.
3) Such systems are easy from maintenance point of view.
4) Generally these are not troubled with the problems of stability.
5) Such systems are simple to design and hence economical.

The good example of an open loop system is an electric switch. This is open loop because output is light and switch is controller of lamp. Any change in light has no effect on the ON-OFF position of the switch, i.e. its controlling action.

Similarly automatic washing machine. Here output is degree of cleanliness of clothes. But any change in this output will not affect the controlling action or will not decide the operation time or will not decide the amount of detergent which is to used. Some other examples are traffic signal, automatic toaster system etc.

5.2 Closed Loop System

Definition : *A system in which the controlling action or input is somehow dependent on the output or changes in output is called closed loop system.*

To have dependence of input on the output, such system uses the feedback property.

Feedback : *Feedback is a property of the system by which it permits the output to be compared with the reference input to generate the error signal based on which the appropriate controlling action can be decided.*

In such system, output or part of the output is fedback to the input for comparison with the reference input applied to it.

Closed loop system can be represented as shown in the Fig. 1.9

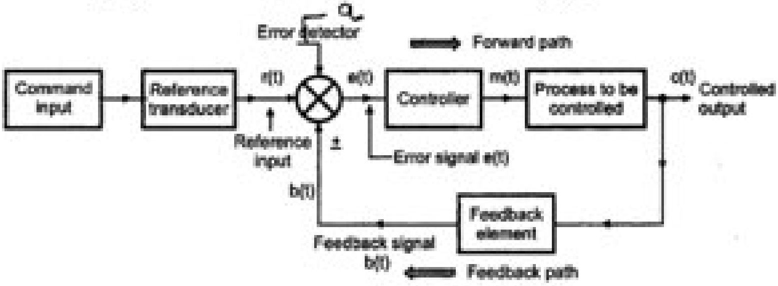


Fig. 1.9 Representation of closed loop control system

The various signals are,

r(t) = Reference input	e(t) = Error signal	
c(t) = Controlled output	m(t) = Manipulated signal	b(t) = Feedback signal

error signal $e(t) = r(t) \pm b(t)$

When feedback sign is positive, systems are called **positive feedback systems** and if it is negative systems are called **negative feedback systems**.

5.2.1 Advantages

The advantages of closed loop system are,

1) Accuracy of such system is always very high because controller modifies and manipulates the actuating signal such that error in the system will be zero.
2) Such system senses environmental changes, as well as internal disturbances and accordingly modifies the error.
3) In such system, there is reduced effect of nonlinearities and distortions.
4) Bandwidth of such system i.e. operating frequency zone for such system is very high.

5.2.2 Home Heating System

In this system, the heating system is operated by a valve. The actual temperature is sensed by a thermal sensor and compared with the desired temperature. The difference between the two, actuates the valve mechanism to change the temperature as per the requirement.

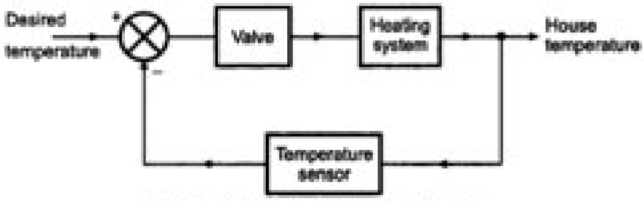


Fig. 1.10 Domestic heating system

5.3 Comparison of Open Loop and Closed Loop Control System

	Open Loop	Closed Loop
1)	Any change in output has no effect on the input i.e. feedback does not exists.	Changes in output, affects the input which is possible by use of feedback.
2)	Output measurement is not required for operation of system.	Output measurement is necessary.
3)	Feedback element is absent.	Feedback element is present.
4)	Error detector is absent.	Error detector is necessary.
5)	It is inaccurate and unreliable.	Highly accurate and reliable.
6)	Highly sensitive to the disturbances.	Less sensitive to the disturbances.
7)	Highly sensitive to the environmental changes.	Less sensitive to the environmental changes.
8)	Bandwidth is small.	Bandwidth is large.
9)	Simple to construct and cheap.	Complicated to design and hence costly.
10)	Generally are stable in nature.	Stability is the major consideration while designing
11)	Highly affected by nonlinearities.	Reduced effect of nonlinearities.

Lesson 2

Basics of Laplace Transform

Basics of Laplace Transform

Objectives:

At the end of this lesson, students will be able to:

1. Use Laplace Transform in solving differential equations of Control System.

Basics of Laplace Transform

The transformation technique relating the time functions to frequency dependent functions of a complex variable is called the **Laplace transformation technique.** Such transformation is very useful in solving linear differential equations.

Definition of Laplace Transform

The Laplace transform is defined as below :

Let f(t) be a real function of a real variable t defined for t > 0, then

$$F(s) = L[f(t)] = \int_0^\infty f(t).e^{-st}\, dt$$

where F(s) is called Laplace transform of f(t). And the variable 's' which appears in F(s) is frequency dependent complex variable.

The time function f(t) is obtained back from the Laplace transform by a process called Inverse Laplace transform and denoted as L^{-1}.

$$L^{-1}[F(s)] = L^{-1}[L(f(t))] = f(t)$$

Example 2.1 : *Find the Laplace transform of e^{-at} and 1 for $t \geq 0$.*

Solution : i) $f(t) = e^{-at}$

$$F(s) = L[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt = \int_0^\infty e^{-at} \cdot e^{-st}\, dt$$

$$= \int_0^\infty e^{-(s+a)t}\, dt = \left[-\frac{1}{(s+a)} \cdot e^{-(s+a)t}\right]_0^\infty$$

$$= 0 - \left(\frac{-1}{s+a}\right) = \frac{1}{s+a}$$

$$\therefore \quad L[e^{-at}] = \frac{1}{s+a} \quad \text{and} \quad L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

(ii) $f(t) = 1$

$$\therefore \quad F(s) = \int_0^\infty f(t)\, e^{-st}\, dt = \int_0^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}$$

$$\therefore \quad L\{1\} = \frac{1}{s} \quad \text{and} \quad L^{-1}\left\{\frac{1}{s}\right\} = 1$$

Properties of Laplace Transform

1. Linearity

So if $F_1(s)$, $F_2(s)$, ..., $F_n(s)$ are the Laplace transforms of the time functions $f_1(t)$, $f_2(t)$, ..., $f_n(t)$ respectively then,

$$L[f_1(t) + f_2(t) + \dots + f_n(t)] = F_1(s) + F_2(s) + \dots + F_n(s)$$

2. Scaling Theorem

$$L\,[K\,f(t)] = K\,F(s)$$

... K is constant

3. Real Differentiation

Let F(s) be the Laplace transform of f(t). Then,

$$L\left\{\frac{d\,f(t)}{dt}\right\} = s\,F(s) - f(0^-)$$

where $f(0^-)$ indicates value of f(t) at $t = 0^-$ i.e. just before the instant $t = 0$

The theorem can be extended for n^{th} order derivative as,

$$L\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \dots - f^{(n-1)}(0^-)$$

where $f^{(n-1)}(0^-)$ is the value of $(n-1)^{th}$ derivative of f(t) at $t = 0^-$.

i.e for $n = 2$, $L\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2 F(s) - s\,f(0^-) - f'(0^-)$

for $n = 3$, $L\left\{\frac{d^3 f(t)}{dt^3}\right\} = s^3 F(s) - s^2 f(0^-) - s f'(0^-) - f''(0^-)$ and so on.

4. Real Integration

If F(s) is the Laplace transform of f(t) then,

$$L\left\{\int_0^t f(t)\,dt\right\} = \frac{F(s)}{s}$$

5. Differentiation by S

$$L\{t\,f(t)\} = \frac{-dF(s)}{ds}$$

Thus,

$$L\{t\} = L\{t\times 1\} = -\frac{d}{ds}[L\{1\}] = -\frac{d}{ds}\left[\frac{1}{s}\right] = \frac{1}{s^2} = \frac{1!}{s^{1+1}}$$

$$L\{t^2\} = L\{t\times t\} = -\frac{d}{ds}[L\{t\}] = -\frac{d}{ds}\left[\frac{1}{s^2}\right] = \frac{2}{s^3} = \frac{2!}{s^{2+1}}$$

6. Complex Translation

$$F(s-a) = L\{e^{at} f(t)\}$$ and $$F(s+a) = L\{e^{-at} f(t)\}$$

$$F(s\mp a) = F(s)\big|_{s=s\mp a}$$

where F(s) is the Laplace transform of f(t).

7. Real Translation (Shifting Theorem)

This theorem is useful to obtain the Laplace transform of the shifted or delayed function of time.

If F(s) is the Laplace transform of f(t) then the Laplace transform of the function **delayed by time T** is,

$$L\{f(t-T)\} = e^{-Ts} F(s)$$

8. Initial Value Theorem

The Laplace transform is very useful to find the initial value of the time function f(t). Thus if F(s) is the Laplace transform of f(t) then,

$$f(0^+) = \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} s\,F(s)$$

9. Final Value Theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$$

Table of Laplace Transforms :

f(t)	F(s)
1	$\frac{1}{s}$
Constant K	$\frac{K}{s}$
K f(t), K is constant	K F(s)
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
e^{at}	$\frac{1}{s-a}$
$e^{-at}\, t^n$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t$	$\frac{(s+a)}{(s+a)^2+\omega^2}$
$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
$t\, e^{-at}$	$\frac{1}{(s+a)^2}$
$1 - e^{-at}$	$\frac{a}{s(s+a)}$

Function f(t)	Laplace Transform F(s)
Unit step = u(t)	$\frac{1}{s}$
A u(t)	$\frac{A}{s}$
Delayed unit step = u(t − T)	$\frac{e^{-Ts}}{s}$
A u(t − T)	$\frac{A\,e^{-Ts}}{s}$
Unit ramp = r(t) = t u(t)	$\frac{1}{s^2}$
A t u(t)	$\frac{A}{s^2}$
Delayed unit ramp = r(t − T) = (t − T) u(t − T)	$\frac{e^{-Ts}}{s^2}$
A (t − T) u(t − T)	$\frac{A\,e^{-Ts}}{s^2}$
Unit impulse = δ(t)	1
Delayed unit impulse = δ(t − T)	e^{-Ts}
Impulse of strength K i.e. K δ(t)	K

Inverse Laplace Transform

As mentioned earlier, inverse Laplace transform is calculated by partial fraction method rather than complex integration evaluation. Let F(s) is the Laplace transform of f(t) then the inverse Laplace transform is denoted as,

$$f(t) = L^{-1}[F(s)]$$

The F(s), in partial fraction method, is written in the form as,

$$F(s) = \frac{N(s)}{D(s)}$$

where N(s) = Numerator polynomial in s

and D(s) = Denominator polynomial in s

The given function F(s) can be expressed in partial fraction form only when **degree of N(s) is less than D(s)**. Hence if degree of N(s) is equal or higher than D(s) then mathematically divide N(s) by D(s) to express F(s) in quotient and remainder form as,

$$F(s) = Q + F_1(s) = Q + \frac{N'(s)}{D'(s)}$$

where Q = Quotient obtained by dividing N(s) by D(s)

and $F_1(s) = \frac{N'(s)}{D'(s)}$ = Remainder

There are 3 types of D(s):

1. Simple and Real Roots

The roots of D(s) are simple and real. Hence the function F(s) can be expressed as,

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s-a)(s-b)(s-c)\ldots}$$

➡ **Example 2.2 :** *Find the inverse Laplace transform of given F(s).*

$$F(s) = \frac{(s+2)}{s(s+3)(s+4)}$$

Solution : The degree of N(s) is less than D(s). Hence F(s) can be expressed as,

$$F(s) = \frac{K_1}{s} + \frac{K_2}{(s+3)} + \frac{K_3}{(s+4)}$$

where $K_1 = s \cdot F(s)\Big|_{s=0} = s \cdot \frac{(s+2)}{s(s+3)(s+4)}\Big|_{s=0} = \frac{2}{3\times 4} = \frac{1}{6}$

$$K_2 = (s+3) \cdot F(s)\Big|_{s=-3} = (s+3) \cdot \frac{(s+2)}{s(s+3)(s+4)}\Big|_{s=-3} = \frac{(-3+2)}{(-3)(-3+4)} = \frac{1}{3}$$

$$K_3 = (s+4) \cdot F(s)\Big|_{s=-4} = (s+4) \cdot \frac{(s+2)}{s(s+3)(s+4)}\Big|_{s=-4} = \frac{(-4+2)}{(-4)(-4+3)} = -\frac{1}{2}$$

$$\therefore \quad F(s) = \frac{1/6}{s} + \frac{1/3}{(s+3)} - \frac{1/2}{(s+4)}$$

Taking inverse Laplace transform,

$$\therefore \quad f(t) = \frac{1}{6} + \frac{1}{3}e^{-3t} - \frac{1}{2}e^{-4t}$$

2. Multiple Roots

The given function is of the form,

$$F(s) = \frac{N(s)}{(s-a)^n \, D'(s)}$$

Example 2.3 : *Obtain the inverse Laplace transform of given F(s).*

$$F(s) = \frac{(s-2)}{s(s+1)^3}$$

Solution : The given F(s) can be expressed as,

$$F(s) = \frac{K_0}{(s+1)^3} + \frac{K_1}{(s+1)^2} + \frac{K_2}{(s+1)} + \frac{K_3}{s}$$

Finding L.C.M. of right hand side,

$$\frac{(s-2)}{s(s+1)^3} = \frac{K_0(s) + K_1(s+1)s + K_2(s+1)^2 s + K_3(s+1)^3}{s(s+1)^3}$$

$$\therefore (s-2) = K_0 s + K_1 s^2 + K_1 s + K_2 s^3 + 2K_2 s^2 + K_2 s + K_3 s^3 + 3K_3 s^2 + 3K_3 s + K_3$$

Comparing coefficients of various powers of s on both sides,

for s^3, $K_2 + K_3 = 0$ ... (1)

for s^2, $K_1 + 2K_2 + 3K_3 = 0$ ... (2)

for s^1, $K_0 + K_1 + K_2 + 3K_3 = 1$ ... (3)

for s^0, $K_3 = -2$ ... (4)

As $K_3 = -2$

from (1), $K_2 = 2$

$\therefore$ from (2), $K_1 = 2$

$\therefore$ from (3), $K_0 = 3$

$$\therefore \quad F(s) = \frac{3}{(s+1)^3} + \frac{2}{(s+1)^2} + \frac{2}{(s+1)} - \frac{2}{s}$$

Now $\quad L\left[e^{-at}\, t^n\right] = \dfrac{n!}{(s+a)^{n+1}}$

$$\therefore \quad L^{-1}\left[\frac{1}{(s+a)^{n+1}}\right] = \frac{e^{-at}\, t^n}{n!}$$

$\therefore \quad F(s) = 3 \cdot \frac{1}{(s+1)^3} + 2 \cdot \frac{1}{(s+1)^2} + 2 \cdot \frac{1}{(s+1)} - 2 \cdot \frac{1}{s}$

$\therefore \quad f(t) = L^{-1}[F(s)] = \frac{3}{2!} e^{-t} \cdot t^2 + \frac{2}{1!} e^{-t} \cdot t + 2e^{-t} - 2$

$\therefore \quad f(t) = \frac{3}{2} t^2 e^{-t} + 2t e^{-t} + 2e^{-t} - 2$

3. Complex Conjugate Roots

If there exists a quadratic term in D(s) of F(s) whose roots are complex conjugates then the F(s) is expressed with a first order polynomial in s in the numerator as,

$$F(s) = \frac{As+B}{(s^2+\alpha s+\beta)} + \frac{N'(s)}{D'(s)}$$

where $(s^2+\alpha s+\beta)$ is the quadratic whose roots are complex conjugates while $\frac{N'(s)}{D'(s)}$ represents remaining terms of the expansion. The A and B are partial fraction coefficients.

Consider $\quad F_1(s) = \frac{As+B}{s^2+\alpha s+\beta} \quad$ A and B are known

Now complete the square in the denominator by calculating last term as,

$$L.T. = \frac{(M.T.)^2}{4\,(F.T.)}$$

where L.T = Last term

M.T = Middle term

F.T = First term

$$\therefore L.T. = \frac{\alpha^2}{4}$$

$$\therefore F_1(s) = \frac{As+B}{s^2+\alpha s+\frac{\alpha^2}{4}+\beta-\frac{\alpha^2}{4}} = \frac{As+B}{\left(s+\frac{\alpha}{2}\right)^2+\omega^2}$$

where $\quad \omega = \sqrt{\beta-\frac{\alpha^2}{4}}$

Now adjust the numerator As + B in such a way that it is of the form,

$$L\left[e^{-at}\sin\omega t\right] = \frac{\omega}{(s+a)^2+\omega^2} \qquad \text{or} \qquad L\left[e^{-at}\cos\omega t\right] = \frac{(s+a)}{(s+a)^2+\omega^2}$$

Key Point: *Thus inverse Laplace transform of F(s) having complex conjugate roots of D(s), always contains sine, cosine or damped sine or damped cosine functions.*

Example 2.4 : *Find the inverse Laplace transform of*

$$F(s) = \frac{s^2+3}{\left(s^2+2s+5\right)(s+2)}$$

Solution : The given F(s) can be written as,

$$F(s) = \frac{As+B}{s^2+2s+5}+\frac{C}{s+2}$$

as s^2+2s+5 has complex conjugate roots. To find A, B and C find L.C.M. of right hand side,

$$\therefore F(s) = \frac{(s+2)(As+B)+C\left(s^2+2s+5\right)}{\left(s^2+2s+5\right)(s+2)}$$

$$\therefore \frac{s^2+3}{\left(s^2+2s+5\right)(s+2)} = \frac{As^2+2As+Bs+2B+Cs^2+2sC+5C}{\left(s^2+2s+5\right)(s+2)}$$

Comparing the coefficients of various powers of s, of the numerators of both sides

$$\therefore s^2+3 = s^2(A+C)+s(2A+B+2C)+(2B+5C)$$

$$\therefore A+C = 1 \qquad \ldots(1)$$

$$\therefore 2A+B+2C = 0 \qquad \ldots(2)$$

$$\therefore 2B+5C = 3 \qquad \ldots(3)$$

To solve the equations quickly, the coefficient C corresponding to the simple, real root can be obtained as,

$$\therefore C = F(s).(s+2)\Big|_{s=-2} = \frac{\left(s^2+3\right)(s+2)}{\left(s^2+2s+5\right)(s+2)}\Bigg|_{s=-2} = \frac{(4+3)}{(4-4+5)} = \frac{7}{5}$$

Substituting in (1) and (2),

$$A = -\frac{2}{5}$$

and $$B = -2$$

$$\therefore F(s) = \frac{-\frac{2}{5}s-2}{s^2+2s+5}+\frac{\frac{7}{5}}{(s+2)}$$

Consider $$F_1(s) = \frac{-\frac{2}{5}s-2}{s^2+2s+5}$$

Completing square in the denominator.

$$F_1(s) = \frac{-\frac{2}{5}s-2}{s^2+2s+1+5-1} = \frac{-\frac{2}{5}s-2}{(s+1)^2+(2)^2} = -\frac{2}{5}\left[\frac{s+5}{(s+1)^2+(2)^2}\right]$$

$$= -\frac{2}{5}\left[\frac{s+1+4}{(s+1)^2+(2)^2}\right] \qquad \text{split 4 as } 2\times 2$$

$$= -\frac{2}{5}\left[\frac{s+1}{(s+1)^2+(2)^2}+2\times\frac{2}{(s+1)^2+(2)^2}\right]$$

$$\therefore F(s) = -\frac{2}{5}\left\{\frac{(s+1)}{(s+1)^2+(2)^2}+2\cdot\frac{2}{(s+1)^2+(2)^2}\right\}+\frac{\frac{7}{5}}{(s+2)}$$

As

$$L^{-1}\left[\frac{(s+a)}{(s+a)^2+\omega^2}\right] = \left[e^{-at}\cos\omega t\right] \quad \text{and}$$

$$L^{-1}\left[\frac{\omega}{(s+a)^2+\omega^2}\right] = \left[e^{-at}\sin\omega t\right]$$

Hence taking inverse Laplace transform of F(s),

$$f(t) = -\frac{2}{5}\left[e^{-t}\cos 2t+2e^{-t}\sin 2t\right]+\frac{7}{5}e^{-2t}$$

Use of Laplace Transform in Control System

Example 2.5 : *Obtain the expression for y(t) which is satisfying the differential equation* $\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 16e^{-t}$. *Neglect initial conditions.*

Solution : Taking Laplace transform of both sides of the given differential equation and neglecting initial condition terms in Laplace transform of $\frac{d^2y(t)}{dt^2}$ and $\frac{dy(t)}{dt}$ we get,

$$s^2Y(s) + 6sY(s) + 8Y(s) = \frac{16}{s+1}$$

$$\therefore \quad (s^2+6s+8)\,Y(s) = \frac{16}{(s+1)}$$

$$\therefore \quad Y(s) = \frac{16}{(s+1)(s^2+6s+8)}$$

$$\therefore \quad Y(s) = \frac{16}{(s+1)(s+2)(s+4)}$$

$$\therefore \quad Y(s) = \frac{a_1}{s+1} + \frac{a_2}{s+2} + \frac{a_3}{s+4}$$

$$\therefore \quad Y(s) = \frac{5.33}{s+1} - \frac{8}{s+2} + \frac{2.66}{s+4}$$

Taking inverse Laplace transform of Y(s),

$$y(t) = 5.33\,e^{-t} - 8\,e^{-2t} + 2.66\,e^{-4t}$$

This is the required solution of differential equation.

Special Case of Inverse Laplace Transform

$$F(s) = \frac{P(s)}{Q(s)} \qquad \text{... order of P(s) and Q(s) same}$$

$$= K + \frac{P'(s)}{Q(s)} \qquad \text{.... after dividing P(s) by Q(s)}$$

Now Laplace inverse of constant term is impulse function. Refer last pair in the Table 2.2

$$\therefore \quad L^{-1}[K] = K\,\delta(t) \qquad \text{where } \delta(t) = \text{unit impulse.}$$

While P' (s) / Q(s) can now be expressed to obtain partial fraction expansion, to get its inverse very easily.

Example 2.6 : *Find the Laplace inverse of* $F(s) = \dfrac{s^3+18s^2+3s+5}{s^3+8s^2+17s+10}$

Solution : Divide P(s) by Q(s).

$$
\begin{array}{r l}
s^3+8s^2+17s+10\;) & s^3+18s^2+\;3s\;+\;5\;(1 \rightarrow K \\
 & s^3+\;8s^2+17s\;+10 \\
 & -\quad -\quad -\quad - \\
\hline
 & 10s^2-14s-5 \;\rightarrow P'(s)
\end{array}
$$

$$\therefore \qquad F(s) = 1+\frac{10s^2-14s-5}{s^3+8s^2+17s+10}$$

$$= 1+\frac{10s^2-14s-5}{(s+2)\,(s+1)\,(s+5)} = 1+\frac{A}{s+2}+\frac{B}{s+1}+\frac{C}{s+5}$$

$$\therefore \qquad A = \left.\frac{10s^2-14s-5}{(s+1)\,(s+5)}\right|_{s=-2} = -21$$

$$B = \left.\frac{10s^2-14s-5}{(s+2)\,(s+5)}\right|_{s=-1} = 4.75$$

$$C = \left.\frac{10s^2-14s-5}{(s+1)\,(s+2)}\right|_{s=-5} = 26.25$$

$$\therefore \qquad F(s) = 1-\frac{21}{s+2}+\frac{4.75}{s+1}+\frac{26.25}{s+5}$$

$$\therefore \qquad f(t) = L^{-1}[F(s)] = \delta(t)-21\,e^{-2t}+4.75\,e^{-t}+26.25\,e^{-5t}$$

where $L^{-1}[1] = \delta(t)$ = Unit impulse function.

Example 2.7 : *Find the inverse Laplace transform of,*

$F(s) = \dfrac{2s+5}{s^2+5s+6}$ **(PTU, Jan.-2006)**

Solution : Factorising denominator,

$$F(s) = \frac{2s+5}{(s+2)(s+3)} = \frac{A}{s+2}+\frac{B}{s+3} \qquad \text{... Partial fractions}$$

$$A = F(s)\,(s+2)|_{s=-2} = \frac{(-4+5)}{(3-2)} = 1$$

$$B = F(s)\,(s+3)|_{s=-3} = \frac{(-6+5)}{(-3+2)} = 1$$

$$\therefore \qquad F(s) = \frac{1}{s+2}+\frac{1}{s+3}$$

$$\therefore \qquad f(t) = L^{-1}[F(s)] = e^{-2t}+e^{-3t}$$

Lesson 3

Mathematical Modelling of Control System

Mathematical Modelling of Control System

Objectives:

At the end of this lesson, students will be able to:

1. Describe a physical system in terms of differential equations

1. Electrical System:

Element	Time domain expression for voltage	Laplace domain expression for voltage	Laplace domain behaviour
Resistance R	$i(t) \times R$	$I(s)R$	R
Inductance L	$L\frac{d\,i(t)}{dt}$	$sLI(s)$	sL
Capacitance C	$\frac{1}{C}\int i(t)\,dt$	$\frac{1}{sC}\,I(s)$	$\frac{1}{sC}$

Table 3.1

Example 3.1 : *Find out the T.F. of the given network.*

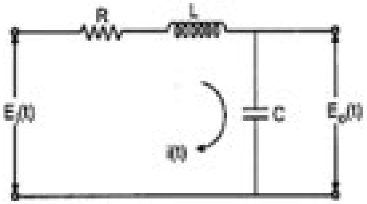


Fig. 3.1

Solution : Applying we get the equations as,

$$E_i = iR + L\frac{di}{dt} + \frac{1}{C}\int i dt \quad \text{-- (1)}$$

$$\text{input} = E_i \quad ; \quad \text{output} = E_o$$

Laplace transform of $\int f(t)\,dt = \frac{F(s)}{s}$, ... neglecting initial conditions

and Laplace transform of $\frac{df(t)}{dt} = sF(s)$ ... neglecting initial conditions

Take Laplace transform,

$$\therefore \quad E_i(s) = I(s)\left[R + sL + \frac{1}{sC}\right]$$

$$\frac{I(s)}{E_i(s)} = \frac{1}{\left[R + sL + \frac{1}{sC}\right]} \quad \text{... (2)}$$

Now
$$E_o = \frac{1}{C}\int i\,dt \quad \text{... (3)}$$

$$\therefore \quad E_o(s) = \frac{1}{sC} I(s)$$

$$\therefore \quad I(s) = sC\, E_o(s) \quad \text{... (4)}$$

Substituting value of I(s) in equation (2),

$$\therefore \quad \frac{sCE_o(s)}{E_i(s)} = \frac{1}{\left[R + sL + \frac{1}{sC}\right]}$$

$$\therefore \quad \frac{E_o(s)}{E_i(s)} = \frac{1}{sC\left[R + sL + \frac{1}{sC}\right]} = \frac{1}{RsC + s^2 LC + 1}$$

So we can represent the system as in the Fig. 3.2 .

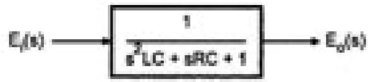
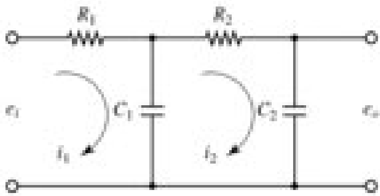


Fig. 3.2 Transfer function model

Transfer Function of Cascaded Elements

Example 3.2: *find out the T.F. of the given network:*



1. Find system equations:

$$\frac{1}{C_1}\int (i_1 - i_2)\,dt + R_1 i_1 = e_i$$

$$\frac{1}{C_1}\int (i_2 - i_1)\,dt + R_2 i_2 + \frac{1}{C_2}\int i_2\,dt = 0$$

$$\frac{1}{C_2}\int i_2\,dt = e_o$$

2. Find Laplace transform:

$$\frac{1}{C_1 s}[I_1(s) - I_2(s)] + R_1 I_1(s) = E_i(s)$$

$$\frac{1}{C_1 s}[I_2(s) - I_1(s)] + R_2 I_2(s) + \frac{1}{C_2 s} I_2(s) = 0$$

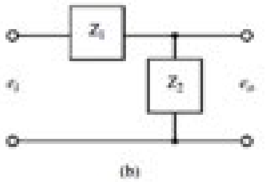
$$\frac{1}{C_2 s} I_2(s) = E_o(s)$$

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{(R_1 C_1 s + 1)(R_2 C_2 s + 1) + R_1 C_2 s}$$

$$= \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2)s + 1}$$

Complex Impedances

Example 3.3: *find out the T.F. of the given network:*



(b)

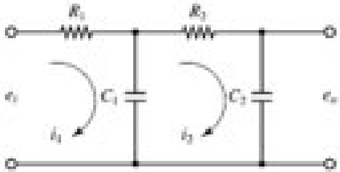
$$Z_1 = Ls + R, \qquad Z_2 = \frac{1}{Cs}$$

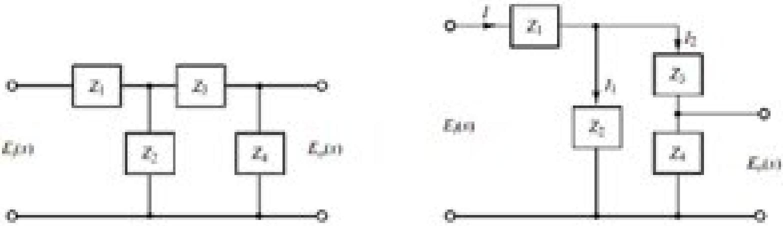
$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$

Hence the transfer function $E_o(s)/E_i(s)$ can be found as follows:

$$\frac{E_o(s)}{E_i(s)} = \frac{\frac{1}{Cs}}{Ls + R + \frac{1}{Cs}} = \frac{1}{LCs^2 + RCs + 1}$$

Example 3.4: *find out the T.F. of the given network:*





(a) (b)

current divider role $$I_1 = \frac{Z_3 + Z_4}{Z_2 + Z_3 + Z_4} I, \qquad I_2 = \frac{Z_2}{Z_2 + Z_3 + Z_4} I$$

$$Z_2 I_1 = (Z_3 + Z_4) I_2, \qquad I_1 + I_2 = I$$

$$E_i(s) = Z_1 I + Z_2 I_1 = \left[Z_1 + \frac{Z_2(Z_3 + Z_4)}{Z_2 + Z_3 + Z_4} \right] I$$

$$E_o(s) = Z_4 I_2 = \frac{Z_2 Z_4}{Z_2 + Z_3 + Z_4} I$$

we obtain

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2 Z_4}{Z_1(Z_2 + Z_3 + Z_4) + Z_2(Z_3 + Z_4)}$$

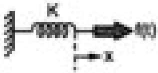
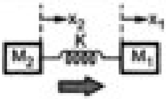
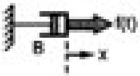
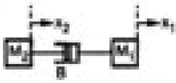
Substituting $Z_1 = R_1$, $Z_2 = 1/(C_1 s)$, $Z_3 = R_2$, and $Z_4 = 1/(C_2 s)$ into this last equation, we get

$$\frac{E_o(s)}{E_i(s)} = \frac{\dfrac{1}{C_1 s} \dfrac{1}{C_2 s}}{R_1\left(\dfrac{1}{C_1 s} + R_2 + \dfrac{1}{C_2 s}\right) + \dfrac{1}{C_1 s}\left(R_2 + \dfrac{1}{C_2 s}\right)}$$

$$= \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2)s + 1}$$

2. Mechanical Systems:

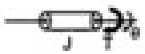
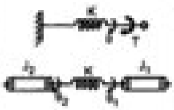
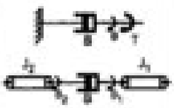
- **Translational Mechanical Systems:**

Element	Type	Representation	Time Equation	Laplace Equation	Remarks
Mass	Translational		$M\frac{d^2x(t)}{dt^2}$	$Ms^2 X(s)$	Always under the influence of single displacement. Can not cause change in displacement as rigid.
Spring	Translational		$K\,x(t)$	$K\,X(s)$	When connected to rigid support, it is under single displacement.
Spring	Translational		$K\,[x_1(t)-x_2(t)]$	$K\,[X_1(s)-X_2(s)]$	When between two moving contacts, causes change in displacement.
Friction	Translational		$B\frac{dx(t)}{dt}$	$B\,s\,X(s)$	When between a fixed support and moving contact, it is under single displacement.
Friction	Translational		$B\frac{d\,[x_1(t)-x_2(t)]}{dt}$	$B\,s\,[X_1(s)-X_2(s)]$	When between two moving contacts, causes change in displacement.

- **Rotational Mechanical Systems:**

Sr. No.	Translational Motion	Rotational Motion
1	Mass (M)	Inertia (J)
2	Friction (B)	Friction (B)
3	Spring (K)	Spring (K)
4	Force (F)	Torque (T)
5	Displacement (x)	Angular displacement (θ)
6	Velocity $v = \left(\frac{dx}{dt}\right)$	Angular velocity $\left(\omega = \frac{d\theta}{dt}\right)$
7	Acceleration $\left(\frac{d^2x}{dt^2}\right)$	Angular acceleration $\left(\alpha = \frac{d^2\theta}{dt^2}\right)$

Table 3.2 Analogous Elements

Inertia	Rotational		$J \frac{d^2\theta(t)}{dt^2}$	$J s^2\theta(s)$	The displacement is angular. Behaviour same as mass in translational.
Spring	Rotational		$K\theta(t)$ $K[\theta_1(t)-\theta_2(t)]$	$K\theta(s)$ $K[\theta_1(s)-\theta_2(s)]$	The spring coefficient K is torsional spring constant.
Friction	Rotational		$B\frac{d\theta(t)}{dt}$ $B\frac{d[\theta_1(t)-\theta_2(t)]}{dt}$	$B s\theta(s)$ $B s[\theta_1(s)-\theta_2(s)]$	The friction coefficient B is torsional friction constant.

2.1 Equivalent Mechanical System (Node Basis)

While drawing analogous networks, it is always better to draw the equivalent mechanical system from the given mechanical system. To draw such system use following steps :

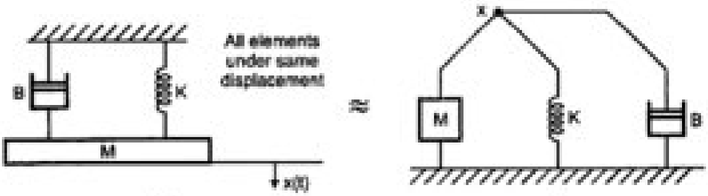
Step 1 : Due to applied force, identify the displacements in the mechanical system.

Step 2 : Identify the elements which are under the influence of different displacements.

Step 3 : Represent each displacement by a separate node, using Nodal Analysis.

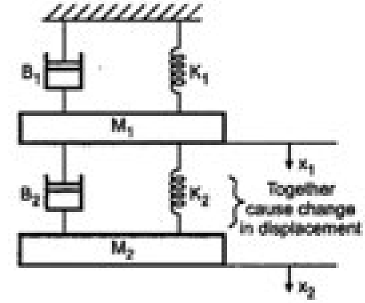
Step 4 : Show all the elements in parallel under the respective nodes which are under the influence of respective displacements.

Step 5 : Elements causing same change in displacement will get connected in parallel in between the respective nodes.



(a) (b)

Fig. 3.3



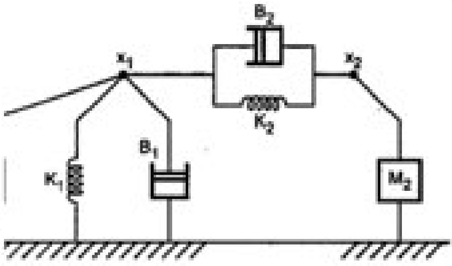


Fig. 3.5

Fig. 3.4

Example 3.5 : *Determine the transfer function* $\frac{Y_2(s)}{F(s)}$ *of the system shown in Fig. 3.6*

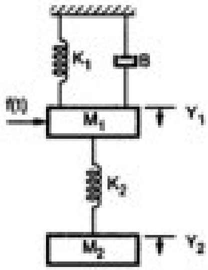


Fig. 3.6

Solution : There are two displacements $y_1(t)$ and $y_2(t)$. The elements M_1, K_1 and B are under the displacement $y_1(t)$ as K_1 and B are between M_1 and fixed support. The element K_2 is between y_1 and y_2, causing change in the displacement.

The element M_2 is under the displacement y_2.

The equivalent mechanical system is as shown.

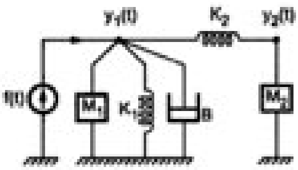


Fig. 3.7 Equivalent system

At node 1, $$f(t) = M_1 \frac{d^2 y_1(t)}{dt^2} + K_1\, y_1(t) + B \frac{d\, y_1(t)}{dt} + K_2 \left[y_1(t) - y_2(t)\right] \quad \text{... (1)}$$

At node 2, $$0 = M_2 \frac{d^2 y_2(t)}{dt^2} + K_2 \left[y_2(t) - y_1(t)\right] \quad \text{... (2)}$$

Taking Laplace transform of both the equations,

$$F(s) = M_1\, s^2\, Y_1(s) + K_1\, Y_1(s) + B\, s\, Y_1(s) + K_2 \left[Y_1(s) - Y_2(s)\right] \quad \text{... (3)}$$

$$0 = M_2\, s^2\, Y_2(s) + K_2 \left[Y_2(s) - Y_1(s)\right] \quad \text{... (4)}$$

From (3), $$F(s) = Y_1(s)\left[M_1\, s^2 + K_1 + K_2 + B\, s\right] - K_2\, Y_2(s)$$

$$\therefore \quad Y_1(s) = \frac{F(s) + K_2\, Y_2(s)}{\left[M_1\, s^2 + B\, s + K_1 + K_2\right]} \quad \text{... (5)}$$

Using in (4), $$0 = M_2\, s^2\, Y_2(s) + K_2\, Y_2(s) - K_2 \left[\frac{F(s) + K_2\, Y_2(s)}{M_1\, s^2 + B\, s + K_1 + K_2}\right]$$

$$\therefore 0 = M_2\, s^2\, Y_2(s) + K_2\, Y_2(s) - \frac{K_2\, F(s)}{(M_1\, s^2 + B\, s + K_1 + K_2)} - \frac{K_2^2\, Y_2(s)}{(M_1\, s^2 + B\, s + K_1 + K_2)}$$

$$\therefore \quad \frac{K_2\, F(s)}{M_1\, s^2 + B\, s + K_1 + K_2} = Y_2(s)\left\{M_2\, s^2 + K_2 - \frac{K_2^2}{(M_1\, s^2 + B\, s + K_1 + K_2)}\right\}$$

$$\therefore \quad K_2\, F(s) = Y_2(s)\left\{(M_2\, s^2 + K_2)(M_1\, s^2 + B\, s + K_1 + K_2) - K_2^2\right\}$$

$$\therefore \quad \frac{Y_2(s)}{F(s)} = \frac{K_2}{(M_2\, s^2 + K_2)(M_1\, s^2 + B\, s + K_1 + K_2) - K_2^2} \quad \text{... Ans}$$

Example 3.6 : *Write the simultaneous differential equations for the translational mechanical system shown in Fig. 3.8 and hence find $X_1(s)$.*

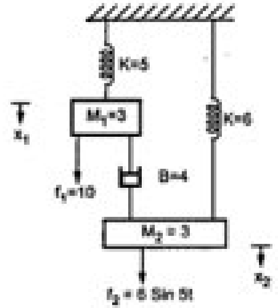


Fig. 3.8

Solution : As seen f_1 is applied to M_1 and f_2 is applied to M_2. The mass M_1 and K = 5 spring are under the displacement x_1. The mass M_2 and K = 6 spring are under the displacement x_2. The friction B = 4 is between x_1 and x_2. Thus the equivalent mechanical system is as shown.

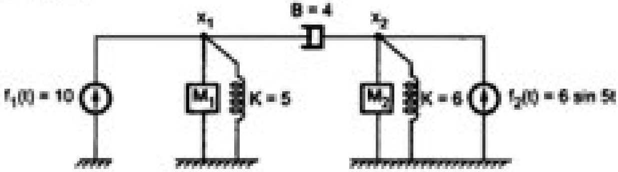


Fig. 3.9

The equilibrium equations are,

$$f_1(t) = M_1 \frac{d^2x_1}{dt^2} + 5x_1 + 4\frac{d[x_1 - x_2]}{dt} \quad ...(1)$$

$$f_2(t) = M_2 \frac{d^2x_2}{dt^2} + 6x_2 + 4\frac{d[x_2 - x_1]}{dt} \quad ...(2)$$

Taking Laplace transform and using the given values,

$$\frac{10}{s} = 3s^2 X_1(s) + 5X_1(s) + 4s\, X_1(s) - 4s\, X_2(s) \quad ...(3)$$

$$\frac{6 \times 5}{s^2 + 5^2} = 3s^2 X_2(s) + 6X_2(s) + 4s\, X_2(s) - 4s\, X_1(s) \quad ...(4)$$

$$\therefore \quad \frac{30}{s^2 + 25} = X_2(s)[3s^2 + 4s + 6] - 4s\, X_1(s)$$

$$\therefore \quad X_2(s) = \frac{\frac{30}{s^2+25} + 4s\, X_1(s)}{3s^2 + 4s + 6} \quad ...(5)$$

Using in (3), $$\frac{10}{s} = X_1(s)\,[3s^2+4s+5] - 4s\left\{\frac{\frac{30}{s^2+25}+4s\,X_1(s)}{3s^2+4s+6}\right\}$$

$$\therefore \quad \frac{10}{s} = X_1(s)\left[3s^2+4s+5-\frac{16s^2}{3s^2+4s+6}\right]-\frac{120\,s}{(s^2+25)(3s^2+4s+6)}$$

$$\therefore \quad \frac{10}{s}+\frac{120\,s}{(s^2+25)(3s^2+4s+6)} = X_1(s)\left\{\frac{(3s^2+4s+5)(3s^2+4s+6)-16s^2}{(3s^2+4s+6)}\right\}$$

$$\therefore \quad \frac{10(s^2+25)(3s^2+4s+6)+120\,s^2}{s(s^2+25)(3s^2+4s+6)} = X_1(s)\left\{\frac{(3s^2+4s+5)(3s^2+4s+6)-16s^2}{(3s^2+4s+6)}\right\}$$

$$\therefore \quad \boxed{X_1(s) = \frac{30\,s^4+40\,s^3+810\,s^2+1000\,s+1500}{s(s^2+25)(9s^4+24s^3+33s^2+44s+30)}}$$

Example 3.7 : *Obtain the transfer function of mechanical system as shown in Fig. 3.10*

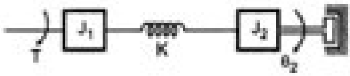


Fig. 3.10

Solution :

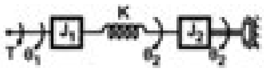


Fig. 3.11 (a)

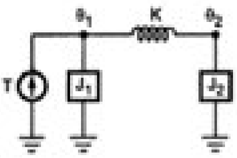


Fig. 3.11 (b)

J_1 is under the displacement θ_1. K is between θ_1 and θ_2 as it causes the change in displacement. J_2 is under the displacement θ_2. The equivalent mechanical system is shown in the Fig. 3.11 (b).

The equations are,

$$T(t) = J_1\frac{d^2\theta_1}{dt^2}+K\,[\theta_1-\theta_2] \qquad \text{... (1)}$$

$$0 = K\,[\theta_2-\theta_1]+J_2\frac{d^2\theta_2}{dt^2} \qquad \text{... (2)}$$

Taking Laplace transform,

$$T(s) = s^2 J_1\,\theta_1(s)+K\theta_1(s)-K\theta_2(s) \qquad \text{... (3)}$$

$$0 = K\theta_2(s)-K\theta_1(s)+J_2\,s^2\,\theta_2(s) \qquad \text{... (4)}$$

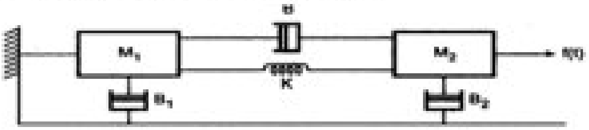
From (4), $$\theta_1(s) = \frac{\theta_2(s)\,[K+J_2\,s^2]}{K}$$

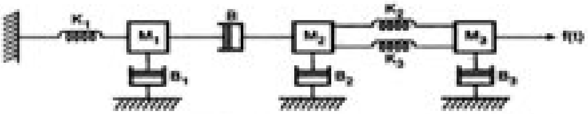
Using in (3), $$T(s) = \theta_2(s)\frac{[K+J_2\,s^2]}{K}\times[J_1\,s^2+K]-K\theta_2(s)$$

$$\therefore \quad T(s) = \theta_2(s)\left[\frac{(K + s^2 J_2)(K + s^2 J_1)}{K} - K\right]$$

$$\therefore \quad \boxed{\frac{\theta_2(s)}{T(s)} = \frac{K}{K s^2 (J_1 + J_2) + s^4 J_1 J_2}} \quad \text{... T.F.}$$

H.W: find T.F. of the following systems





Lesson 4

Servomotors

Servomotors

Objectives:

At the end of this lesson, students will be able to:

1. Derive T.F. for describing the work of servomotors.

Servomotors

These motors are used to convert electrical signal applied, into the angular velocity or movement of shaft.

Types of servomotors:

The types of servomotors are as shown in the following chart :

1. **DC Servomotor**
   a. **Field Controlled DC Servomotor**

Features of Field Controlled D.C. Servomotor

It has following features :

i) Preferred for small rated motors.

ii) It has large time constant.

iii) It is open loop system. This means any change in output has no effect on the input.

iv) Control circuit is simple to design.

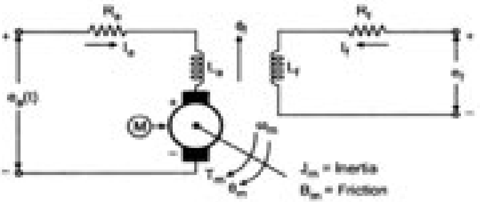


Fig. 4.1

Assumptions :

(1) Constant armature current is fed into the motor.

(2) $\phi_f \propto I_f$. Flux produced is proportional to field current.

$$\therefore \quad \phi_f = K_f I_f$$

(3) Torque is proportional to product of flux and armature current.

$$T_m \propto \phi I_a$$

$\therefore \quad T_m = K' \phi I_a = K' K_f I_f I_a$

$$\boxed{T_m = K_m K_f I_f} \quad \text{... (1)}$$

Where $K_m = K' I_a$ = Constant

Apply Kirchhoff's law to field circuit.

$$L_f \frac{di_f}{dt} + R_f I_f = e_f \quad \text{... (2)}$$

Now shaft torque T_m is used for driving load against the inertia and frictional torque.

$$T_m = J_m \frac{d^2\theta_m}{dt^2} + B_m \frac{d\theta_m}{dt} \quad \text{... (3)}$$

$$\text{Inertia force} = J_m \frac{d^2\theta_m}{dt^2} \text{ similar to } m \frac{d^2x}{dt^2}$$

$$\text{Frictional force} = B_m \frac{d\theta_m}{dt} \text{ similar to } B \frac{dx}{dt}$$

Finding Laplace Transforms of equations (1), (2) and (3) we get.

$$T_m(s) = K_m K_f I_f(s) \quad \text{... (4)}$$

$$E_f(s) = (sL_f + R_f)\ I_f(s) \quad \text{... (5)}$$

$$T_m(s) = J_m s^2 \theta_m(s) + B_m s\, \theta_m(s) \quad \text{... (6)}$$

Eliminate $I_f(s)$ from equations (4) and (5)

$$T_m(s) = \frac{K_m K_f E_f(s)}{(sL_f + R_f)} \quad \text{... (7)}$$

Eliminate $T_m(s)$ from equations (6) and (7).

$$(s^2 J_m + sB_m)\,\theta_m(s) = \frac{K_m K_f E_f(s)}{(sL_f + R_f)}$$

Input = $E_f(s)$

Output = Rotational displacement $\theta_m(s)$

$$\therefore \text{ Transfer function } = \frac{\theta_m(s)}{E_f(s)}$$

$$\frac{\theta_m(s)}{E_f(s)} = \frac{K_m K_f}{(J_m s^2 + sB_m)\ (R_f + sL_f)}$$

$$= \frac{K_m K_f}{sR_f B_m [1 + s\tau_m][1 + s\tau_f]}$$

Where $\tau_m = \dfrac{J_m}{B_m}$ = Motor time constant

$\tau_f = \dfrac{L_f}{R_f}$ = Field time constant

$$\text{T.F.} = \frac{\theta_m(s)}{E_f(s)} = \frac{K_f}{R_f(1+s\tau_f)} \cdot \frac{K_m}{B_m(1+s\tau_m)} \cdot \frac{1}{s}$$

Block diagram for field controlled d.c. motor is as shown in Fig. 4.3 4.2

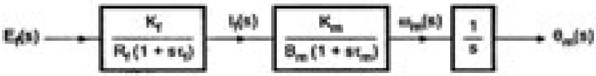


Fig. 4.2 Block diagram

b. Armature Controlled DC Servomotor

1 Features of Armature Controlled D.C. Servomotor

It has following features :

i) Suitable for large rated motors.

ii) It has small time constant hence its response is fast to the control signal.

iii) It is closed loop system.

iv) The back e.m.f. provides internal damping which makes motor operation more stable.

v) The efficiency and overall performance is better than field controlled motor.

As the armature controlled d.c. servomotor is closed loop system, in comparison with open loop field controlled system, generally armature controlled motors are used.

Assumptions :

(i) Flux is directly proportional to current through field winding.

$$\phi_m = K_f I_f = \text{Constant}$$

(ii) Torque produced is proportional to product of flux and armature current.

$$T = K'_m \phi I_a$$

$$T = K'_m K_f I_f I_a$$

(iii) Back e.m.f. is directly proportional to shaft velocity ω_m, as flux ϕ is constant.

$$\text{as} \quad \omega_m = \frac{d\theta(t)}{dt}$$

$$E_b = K_b \omega_m(s) = K_b s \theta_m(s)$$

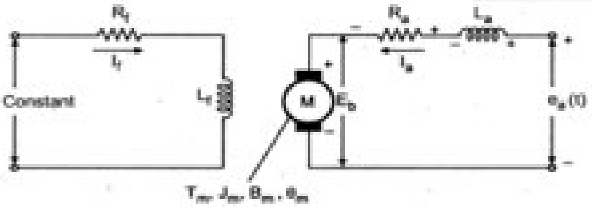


Fig. 4.3

Apply Kirchhoff's law to armature circuit :

$$e_a = E_b + I_a(R_a) + L_a \frac{di_a}{dt}$$

Take Laplace transform,

$$\therefore \quad E_a(s) = E_b(s) + I_a(s)[R_a + sL_a]$$

$$\therefore \quad I_a(s) = \frac{E_a(s) - E_b(s)}{R_a + sL_a}$$

$$I_a(s) = \frac{E_a(s) - K_b\, s\theta_m(s)}{R_a + sL_a}$$

Now

$$T_m = K'_m K_f I_f I_a$$

$$T_m = K'_m K_f I_f \left\{\frac{E_a - K_b\, s\theta_m(s)}{R_a + sL_a}\right\}$$

Also

$$T_m = (J_m s^2 + sB_m)\theta_m(s) \qquad \text{... from equation (3)}$$

Equating equations of T_m,

$$\frac{K'_m K_f I_f E_a(s)}{(R_a + sL_a)} = \frac{K'_m K_f I_f K_b\, s\theta_m(s)}{(R_a + sL_a)} + (J_m s^2 + sB_m)\theta_m(s)$$

$$\therefore \quad \frac{K'_m K_f I_f}{(R_a + sL_a)} E_a(s) = \left[\frac{K'_m K_f I_f K_b\, s}{(R_a + sL_a)} + J_m s^2 + sB_m\right]\theta_m(s)$$

$$\therefore \quad \boxed{\frac{\theta_m(s)}{E_a(s)} = \frac{\dfrac{K_m}{sR_a B_m(1+s\tau_m)(1+s\tau_a)}}{1 + \dfrac{K_m \cdot sK_b}{sR_a B_m(1+s\tau_m)(1+s\tau_a)}} = \frac{G(s)}{1+G(s)H(s)}}$$

where $\tau_m = J_m/B_m$ and $\tau_a = \dfrac{L_a}{R_a}$

$$K_m = K'_m K_f$$

$$G(s) = \frac{K_m}{sR_a B_m(1+s\tau_m)(1+s\tau_a)}$$

$$H(s) = sK_b$$

Therefore can be represented in its block diagram form as in Fig. 4.4

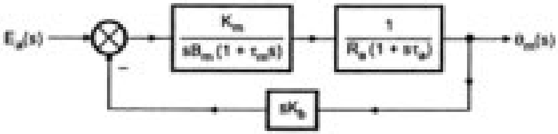


Fig. 4.4 Block diagram

Key Point : *Field controlled d.c. motor is open loop while armature controlled is closed loop system. Hence armature controlled d.c. motors are preferred over field controlled type.*

1. AC Servomotor

Construction

It is mainly divided into two parts namely stator and rotor.

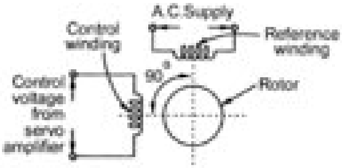


Fig. 4.5 Stator of A.C. servomotor

The stator carries two windings, uniformly distributed and displaced by 90°, in space. One winding is called **main winding** or **fixed winding** or **reference winding**. This is excited by a constant voltage a.c. supply. The other winding is called **control winding**. It is excited by variable control voltage, which is obtained from a servoamplifier. This voltage is 90° out of phase with respect to the voltage applied to the reference winding. This is necessary to obtain rotating magnetic field. The schematic stator is shown in the Fig 4.5

Lesson 5

Block Diagram Reduction

Block Diagram Reduction

Objectives:

At the end of this lesson, students will be able to:

1. Reduce a block diagram of multiple subsystems to a single block representing the T.F. of the system.

Block Diagram Representation

Signal block Take off point summing point

Closed-Loop Control System

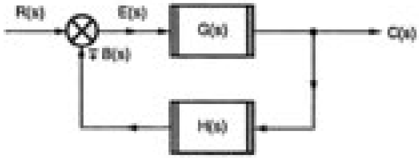


Fig. 5.1

where , $R(s)$ → Laplace of reference input $r(t)$

$C(s)$ → Laplace of controlled output $c(t)$

$E(s)$ → Laplace of error signal $e(t)$

$B(s)$ → Laplace of feedback signal $b(t)$

$G(s)$ → Equivalent forward path transfer function

$H(s)$ → Equivalent feedback path transfer function.

5.1 Derivation of T.F. of Simple Closed Loop System

Referring to the Fig. 5.1, we can write following equations as,

$$E(s) = R(s) \mp B(s) \quad \text{... (1)}$$

$$B(s) = C(s)H(s) \quad \text{... (2)}$$

$$C(s) = E(s)G(s) \quad \text{... (3)}$$

$B(s) = C(s)H(s)$ and substituting in equation (1)

$$E(s) = R(s) \mp C(s)H(s)$$

$$E(s) = \frac{C(s)}{G(s)}$$

$$\frac{C(s)}{G(s)} = R(s) \mp C(s)H(s)$$

$$C(s) = R(s)G(s) \mp C(s)G(s)H(s)$$

$$\therefore C(s)\,[1 \pm G(s)\,H(s)\,] = R(s)G(s)$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$

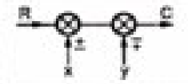
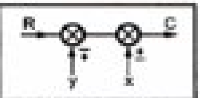
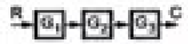
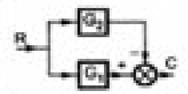
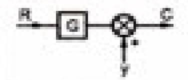
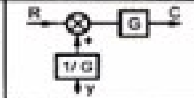
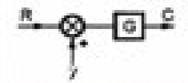
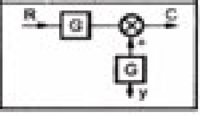
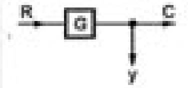
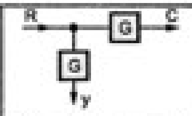
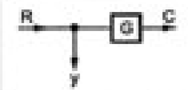
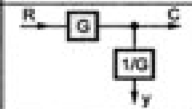
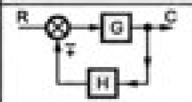
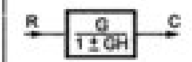
Use + sign for negative feedback and Use − sign for positive feedback.

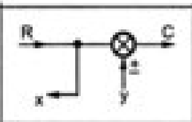
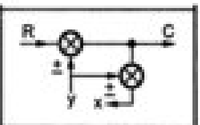
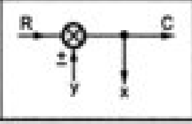
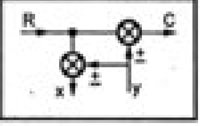
This can be represented as in the Fig. 5.2

Closed Loop T.F.

Fig. 5.2

Table of Block Diagram Reduction Rules

1.	Associative Law	The two or more summing points directly connected can be interchanged.		
2.	Blocks in series	Transfer functions of such blocks get multiplied		
3.	Blocks in parallel	Transfer functions of such blocks get added algebraically		
4.	Shifting summing point behind the block	Add a block of T.F. equal to reciprocal of block behind which summing point is to be shifted in series with all signals at that summing point		
5.	Shifting summing point beyond the block	Add a block of T.F. same as the block beyond which summing point is to be shifted, in series with all the signals at that summing point		
6.	Shifting a take off point behind the block	Add a block of T.F. equal to the block behind which take off point is to be shifted, in series with all the signals at that take off point		
7.	Shifting a take off point beyond the block	Add a block of T.F. equal to the reciprocal of the block beyond which take off point is to be shifted, in series with all the signals at that take off point		
8.	Removing minor feedback loop	Use the standard T.F. of a simple closed loop system		

9.	Shifting take off point after summing point	Add a new summing point to compensate for the shift as shown in example		
10.	Shifting a take off point before a summing point	Add a summing point to compensate for a shift of take off point		

Rule 11: for multiple input system, use super position theorem

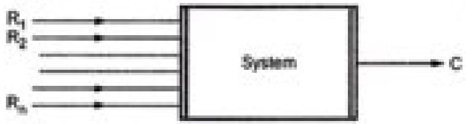


Fig. 5.3

Consider only one input at a time treating all other as zero.

Consider R_1, $R_2 = R_3 = \ldots\ldots R_n = 0$ and find output C_1,

Then consider R_2, $R_1 = R_3 = \ldots\ldots R_n = 0$ and find output C_2

At the end when all inputs are covered, take algebraic sum of all the outputs.

Total output $C = C_1 + C_2 + \ldots\ldots C_n$

5.2 Procedure to Solve Block Diagram Reduction Problems

Step 1 : Reduce the blocks connected in series.

Step 2 : Reduce the blocks connected in parallel.

Step 3 : Reduce the minor internal feedback loops.

Step 4 : As far as possible try to shift take off points towards right and summing points to the left. Unless and until it is the requirement of problem do not use rule 10 and 11.

Step 5 : Repeat steps 1 to 4 till simple form is obtained.

Step 6 : Using standard T.F. of simple closed loop system, obtain the closed loop T.F. $\frac{C(s)}{R(s)}$ of the overall system.

➡ **Example 5.1 :** *Determine the transfer function $C(s)/R(s)$ of the system shown in the Fig. 5.4* **[AU : April-2004, Dec.-2004]**

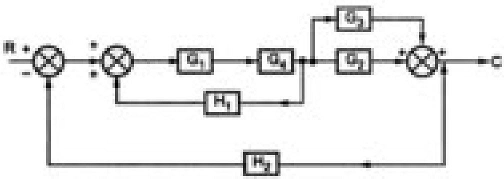
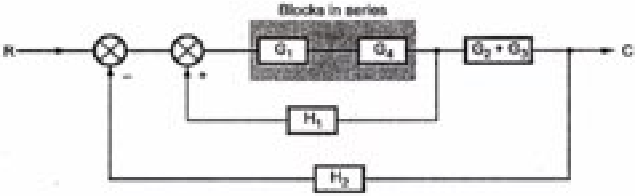
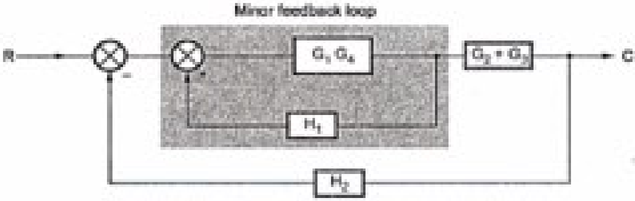
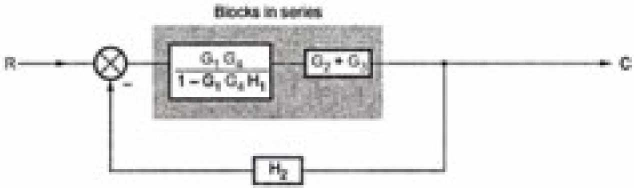


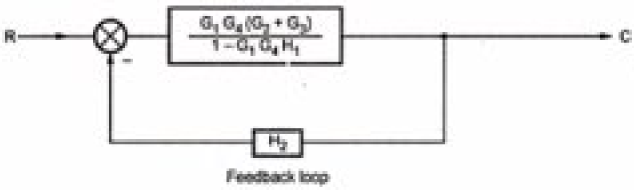
Fig. 5.4

Solution : The blocks G_2 and G_3 are in parallel so combining them as (G_2+G_3) we get.







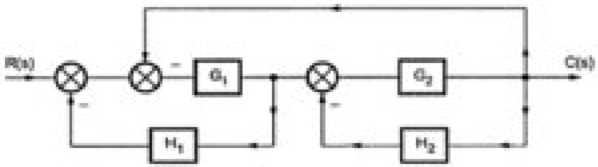


$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\dfrac{G_1 G_4 (G_2+G_3)}{1-G_1 G_4 H_1}}{1+\dfrac{G_1 G_4 (G_2+G_3) H_2}{1-G_1 G_4 H_1}}$$

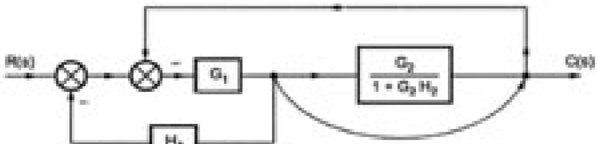
$$\therefore \quad \boxed{\frac{C(s)}{R(s)} = \frac{G_1 G_4 (G_2+G_3)}{1-G_1 G_4 H_1 + G_1 G_4 (G_2+G_3) H_2}} \quad \text{... Ans}$$

Example 5.2 : *Reduce the block diagram and obtain its closed loop T.F. C(s)/R(s).*

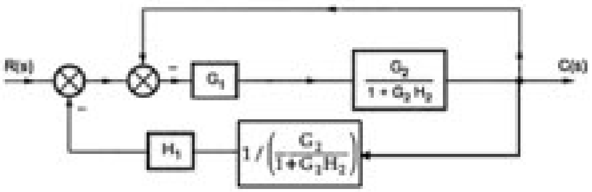
[AU : May-2009]

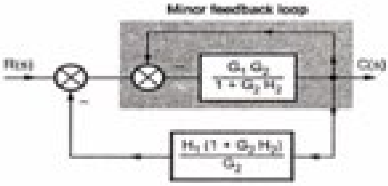


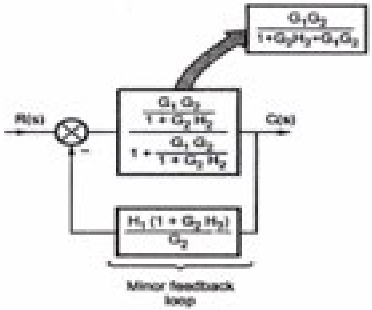
Solution : No blocks are connected in series or parallel. Blocks having transfer functions G_2 and H_2 form minor feedback loop so eliminating that loop we get.

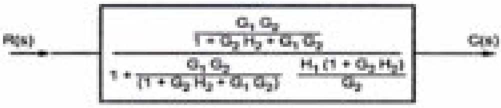


Key Point: *Always try to shift take off point towards right i.e. output side and summing point towards left i.e. input side.*



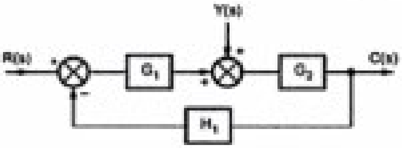




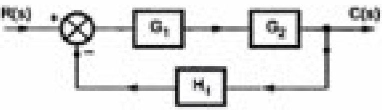


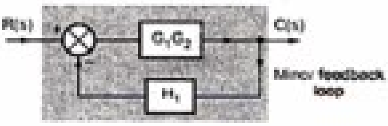
Simplifying,
$$\frac{C(s)}{R(s)} = \frac{G_1\, G_2}{1+G_1\, G_2+G_2\, H_2+G_1\, H_1+G_1\, G_2\, H_1\, H_2}$$

➡ **Example** 5.3 : *Obtain the resultant output C(s) in terms of the inputs R(s) and Y(s).*



Solution : As there are two inputs, consider each input separately. **Consider R(s),** assuming Y(s) = 0.





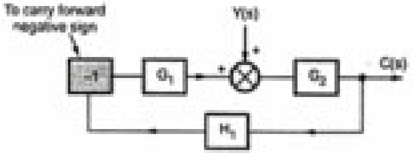
$$\frac{C(s)}{R(s)} = \frac{G_1\, G_2}{1+G_1\, G_2\, H_1}$$

So part of C(s) due to R(s) alone is,

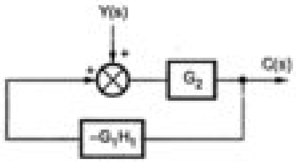
$$C(s) = R(s)\left[\frac{G_1 G_2}{1+G_1 G_2 H_1}\right]$$

Now consider Y(s) acting with R(s) = 0.

Now sign of signal obtained from H_1 is negative which must be carried forward, though summing point at R(s) is removed, as R(s) = 0, so we get,



Combining the blocks G_1 H_1 and −1 as in series,



Key Point: *While finding equivalent G, trace forward path from input summing point to output in direction of signal. While finding equivalent H, trace the feedback path from output to input summing point in the direction of signal.*

Now equivalent G = G_2, tracing forward path from input summing point to output.

Equivalent H = $-G_1 H_1$ tracing feedback path from output to input summing point.

While sign of the final feedback is positive at the input summing point.

$$\therefore \quad \frac{C(s)}{Y(s)} = \frac{G}{1-GH} = \frac{G_2}{1-G_2(-G_1 H_1)}$$

H itself is negative.

$$\therefore \quad \frac{C(s)}{Y(s)} = \frac{G_2}{1+G_1 G_2 H_1}$$

So part of C(s) due to Y(s) alone is,

$$C(s) = Y(s)\left[\frac{G_2}{1+G_1 G_2 H_1}\right]$$

Hence the net output C(s) is given by algebraically adding its two components,

$$C(s) = \frac{G_1 G_2 R(s) + G_2 Y(s)}{1+G_1 G_2 H_1}$$

Key Point: *Students can remember as a crosscheck that while finding the ratio of output with each input, the denominator remains same. For example in the above problem it is* $1+G_1G_2H_1$, *for both the ratios* $\frac{C(s)}{R(s)}$ *and* $\frac{C(s)}{Y(s)}$.

Lesson 6

Time Response Analysis of Control Systems

Time Response Analysis of Control Systems

Objectives:

At the end of this lesson, students will be able to:

1. Understand steady state and transient time response analysis.
2. Find the 1st order system time response.
3. Find the 2nd order system time response.

Time Response Analysis of Control Systems

Definition : ***Time Response :*** The response given by the system which is function of the time, to the applied excitation is called as Time Response of a control system.

Hence total time response C(t) we can write as,

$$C(t) = C_{ss}(t) + C_t(t)$$

Definition : ***Transient Response*** :

The output variation during the time, it takes to achieve its final value is called as Transient Response. The time required to achieve the final value is called as **transient period.**

The transient response may be exponential or oscillatory in nature. Symbolically it is denoted as $C_t(t)$.

Mathematically for stable operating systems,

$$\lim_{t \to \infty} C_t(t) = 0$$

Definition : ***Steady State Response*** :

It is that part of the time response which remains after complete transient response vanishes from the system output. The symbol for steady state output is $C_{ss}(t)$.

The above definitions can be shown in the waveform as in Fig. 6.1 (a), (b) where input applied to the system is step type of input.

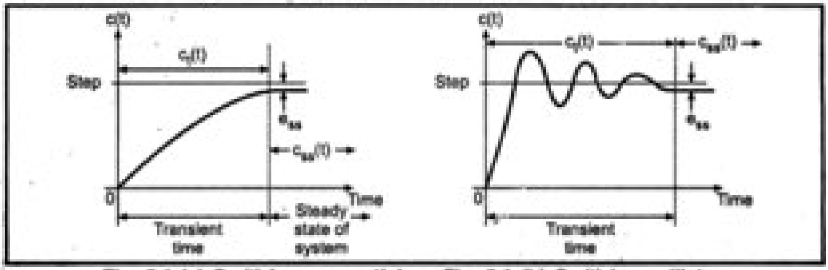


Fig. 6.1 (a) C_t (t) is exponential **Fig. 6.1 (b) C_t (t) is oscillatory**

Standard Test Inputs

i) Step Input (Position function) :

Mathematically it can be described as,

$$R(t) = A \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$

If A = 1, then it is called as **unit step function** and denoted by u(t).

Laplace transform of such input is $\frac{A}{s}$.

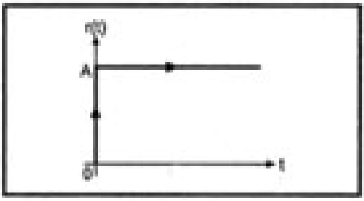


Fig. 6.2

ii) Ramp Input (Velocity function) :

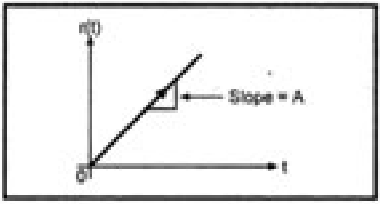


Fig. 6.3

$$R(t) = At \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$

If A = 1, it is called as **Unit Ramp input**. Its Laplace transform is $\frac{A}{s^2}$.

iii) Parabolic Input (Acceleration Function) :

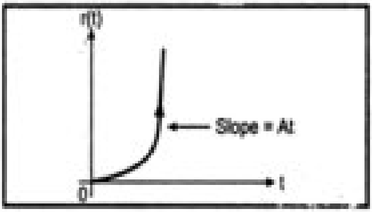


Fig. 6.4

$$R(t) = \frac{A}{2}t^2, \quad \text{for } t \geq 0$$
$$= 0, \quad \text{for } t < 0$$

Function is expressed as $\frac{A}{2}t^2$ so that in Laplace transforms of different standard inputs similarity will get maintained. If A = 1, i.e. $R(t) = \frac{t^2}{2}$ it is called as **unit parabolic input**. Its Laplace transform is $\frac{A}{s^3}$.

iv) Impulse Input :

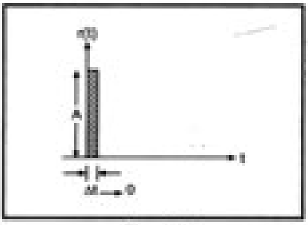


Fig. 6.5

Mathematically it can be expressed as,

$$R(t) = A, \quad \text{for } t = 0$$
$$= 0, \quad \text{for } t \neq 0$$

Its Laplace transform is always 1 if A = 1. i.e. for unit impulse response.
The unit impuse is denoted as δ(t)

In general its Laplace transform is 'A' which is constant.

R(t)	R(s)
Unit step	1/s
Unit ramp	$1/s^2$
Unit parabolic	$1/s^3$
Unit impulse	1

Table 6.1

Steady State Analysis

Definition : ***Steady State Error*** *: It is the difference between the actual output and the desired output.*

Consider a simple closed loop system using negative feedback as shown in Fig. 6.6
where E(s) = Error signal, and B(s) = Feedback signal

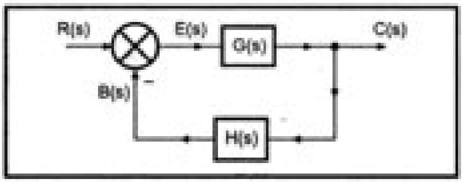


Fig. 6.6

Now, $E(s) = R(s) - B(s)$

But $B(s) = C(s) \cdot H(s)$

∴ $E(s) = R(s) - C(s)\,H(s)$

and $$C(s) = E(S) \cdot G(s)$$

$\because$ $$E(s) = R(s) - E(s)\, G(s)\, H(s)$$

$\therefore$ $$E(s) + E(s)\, G(s)\, H(s) = R(s)$$

$\therefore$ $$E(s) = \frac{R(s)}{1 + G(s)\, H(s)} \quad \text{for nonunity feedback}$$

$$E(s) = \frac{R(s)}{1 + G(s)} \quad \text{for unity feedback}$$

Now we can relate this in Laplace domain by using final value theorem which states that,

$$\lim_{t \to \infty} F(t) = \lim_{s \to 0} s\, F(s) \qquad \text{where } F(s) = L\,[\,F(t)\,]$$

Therefore, $$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} s\, E(s) \qquad \text{where } E(s) \text{ is } L\,[\,e(t)\,].$$

Substituting E(s) from the expression derived, we can write

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)\, H(s)}$$

From the above expression it can be concluded that steady state error depends on,

i) R(s) i.e. reference input, its type and magnitude.

ii) G(s) H(s) i.e. open loop transfer function.

iii) Dominant nonlinearities present if any.

Effect of Input on Steady State Error (static Error Coefficient Method)

Consider a system having open loop T.F. G(s) H(s) and excited by,

a) Reference input is step of magnitude A :

$$R(s) = \frac{A}{s}$$

$\therefore$ $$e_{ss} = \lim_{s \to 0} \frac{s\, R(s)}{1 + G(s)\, H(s)}$$

$$= \lim_{s \to 0} \frac{s \cdot A/s}{1 + G(s)\, H(s)}$$

$$= \lim_{s \to 0} \frac{A}{1 + G(s)\, H(s)}$$

$\therefore$ $$e_{ss} = \frac{A}{1 + \lim_{s \to 0} G(s)\, H(s)}$$

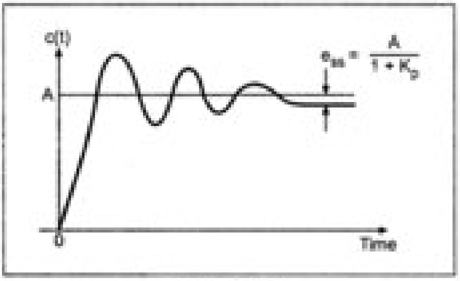


Fig. 6.7

$$K_p = \lim_{s \to 0} G(s)\,H(s) = \text{positional error coefficient and}$$

corresponding error is,

$$e_{ss} = \frac{A}{1 + K_p}$$

So whenever step input is selected as a reference input, positional error coefficient K_p will control the error in the system alongwith the magnitude of the input applied.

b) Reference input is ramp of magnitude 'A' :

$$R(s) = A/s^2$$

$$\therefore \quad e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)\,H(s)}$$

$$= \lim_{s \to 0} \frac{s \cdot A/s^2}{1 + G(s)\,H(s)}$$

$$= \lim_{s \to 0} \frac{A}{s\,[1 + G(s)\,H(s)]}$$

$$= \lim_{s \to 0} \frac{A}{s + s\,G(s)\,H(s)}$$

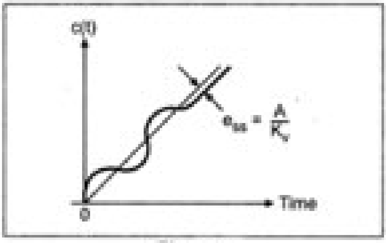


Fig. 6.8

$$\therefore \quad e_{ss} = \frac{A}{\lim_{s \to 0} s\,G(s)\,H(s)}$$

$$K_v = \lim_{s \to 0} s\,G(s)\,H(s) = \text{Velocity error coefficient}$$

and corresponding error is,

$$e_{ss} = \frac{A}{K_v}$$

So whenever ramp input is selected as a reference input, velocity error coefficient K_v will control the error in the system alongwith the magnitude of input applied.

c) Reference input is parabolic of magnitude 'A' :

i.e. $$R(t) = \frac{A}{2} t^2$$

$$\therefore \quad R(s) = \frac{A}{s^3}$$

$$\therefore \quad e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)\,H(s)}$$

$$= \lim_{s \to 0} \frac{s \cdot A/s^3}{1 + G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{A}{s^2[1 + G(s)H(s)]}$$

$$= \lim_{s \to 0} \frac{A}{s^2 + s^2 G(s)H(s)}$$

$$\therefore \quad e_{ss} = \frac{A}{\lim_{s \to 0} s^2 G(s)H(s)}$$

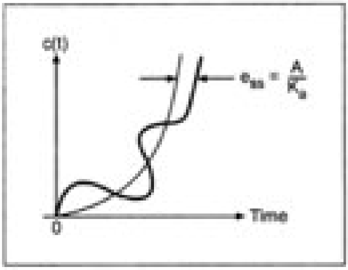


Fig. 6.9

$$\therefore \quad K_a = \lim_{s \to 0} s^2 G(s)H(s) = \text{Acceleration error coefficient}$$

and corresponding error is,

$$e_{ss} = \frac{A}{K_a}$$

So whenever parabolic input is selected as a reference input, acceleration error coefficient K_a will control the error in the system alongwith magnitude of input applied. So static error coefficients are given in Table 6.2

Static Error Coefficients
$K_p = \lim_{s \to 0} G(s)H(s)$
$K_v = \lim_{s \to 0} s\,G(s)H(s)$
$K_a = \lim_{s \to 0} s^2 G(s)H(s)$

Table 6.2

Effect of Change in G(s) H(s) on Steady State Error (TYPE of a System)

Analysis of TYPE 0, 1 and 2 Systems

Note : A popular method to assess steady state performance of servomechanisms or unity feedback systems is to find their error co-efficients K_p, K_v and K_a.

$$G(s)H(s) = \frac{K(1 + T_1 s)(1 + T_2 s)\ldots}{s^j(1 + T_a s)(1 + T_b s)\ldots}$$

where K = Resultant system gain and j = TYPE of the system

'TYPE' of the system means number of poles at origin of open loop T.F. G(s) H(s) of the system.

So $j = 0$, TYPE zero system

$j = 1$, TYPE one system

$j = 2$, TYPE two system

:

:

$j = n$, TYPE 'n' system

Consider the input selected as step of magnitude 'A'.

i) Let us assume that the system is of TYPE '0'.

i.e. $$G(s)\,H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots}{(1+T_a\, s)(1+T_b\, s)\ldots\ldots}$$

For step input $K_p = \lim_{s \to 0} G(s)\,H(s) = K$ ... using above G(s) H(s)

$$\therefore \quad e_{ss} = \frac{A}{1+K_p} = \frac{A}{1+K}$$

i.e. TYPE '0' systems follow the step type of input with finite error $\frac{A}{1+K}$ which

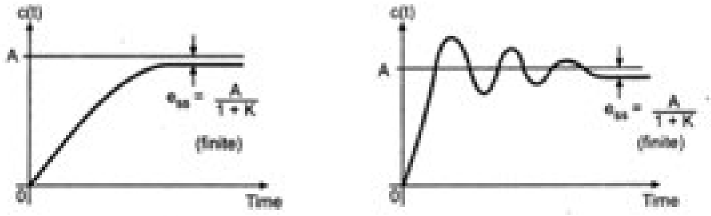


Fig. 6.10 (a) Fig. 6.10 (b)

ii) If for the same input now TYPE is increased to 'one' by adding pole at origin in G(s) H(s).

TYPE 1 : $$G(s)\,H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots}{s(1+T_a s)(1+T_b s)\ldots\ldots}$$

As input is step , $K_p = \lim_{s \to 0} G(s)\; H(s) = \infty$

$$\therefore \quad e_{ss} = \frac{A}{1+K_p} = \frac{A}{\infty} = 0$$

iii) Similarly if now TYPE is further increased to 'two' i.e. G(s) H(s) with 2 poles at origin,

TYPE 2 :

$$G(s)\,H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots}{s^2(1+T_a s)(1+T_b s)\ldots\ldots}$$

As input is step,

$$K_p = \lim_{s\to 0} G(s)\;H(s) = \infty$$

$$e_{ss} = \frac{A}{1+K_p} = \frac{A}{\infty} = 0$$

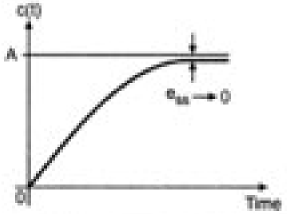


Fig. 6.11 (a)

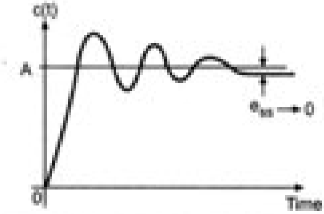


Fig. 6.11 (b)

Let us now change the selected input from step to ramp of magnitude 'A' so K_v will control the error .

iv) Let the system be of TYPE 0.

$$G(s)\,H(s) = \frac{K(1+T_1\ s)\,(1+T_2\ s)\ \ldots}{(1+T_a\ s)\,(1+T_b\ s)\ldots\ldots}$$

$$\therefore \qquad K_v = \lim_{s\to 0} s\,G(s)\,H(s) = 0 \qquad \therefore \quad e_{ss} = \frac{A}{K_v} = \frac{A}{0} = \infty$$

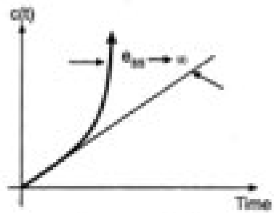


Fig. 6.12 (a)

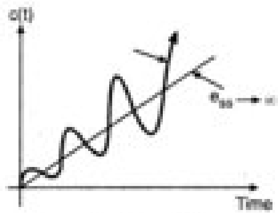


Fig. 6.12 (b)

v) If TYPE 1 System is subjected to Ramp input then,

$$G(s)\,H(s) = \frac{K(1+T_1 s)(1+T_2\ s)\ldots\ldots}{s(1+T_a\ s)(1+T_b\ s)\ldots\ldots}$$

$$\therefore \quad K_v = \lim_{s\to 0} s\,G(s)\,H(s) = K \qquad \therefore e_{ss} = \frac{A}{K_v} = \frac{A}{K}\ \text{finite}$$

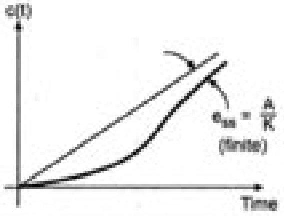


Fig. 6.13 (a)

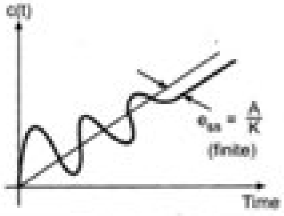


Fig. 6.13 (b)

vi) If TYPE 2 system is excited by Ramp input then,

i.e. $$G(s)\,H(s) = \frac{K(1+T_1\ s)\ (1+T_2\ s)\ldots\ldots}{s^2(1+T_a\ s)\ (1+T_b\ s)\ldots\ldots}$$

$$\therefore \quad K_v = \lim_{s\to 0} s\,G(s)\ H(s) = \infty \qquad \therefore \quad e_{ss} = \frac{A}{K_v} = \frac{A}{\infty} = 0$$

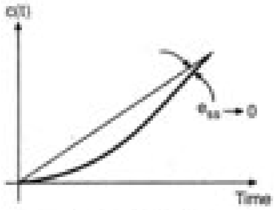


Fig. 6.14 (a)

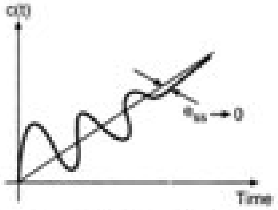


Fig. 6.14 (b)

Let us now change the selected input from ramp to parabolic input of magnitude A hence coefficient K_a will control the error.

vii) Consider TYPE 0 system :

$$G(s)\,H(s) = \frac{K(1+T_1\ s)\ (1+T_2\ s)\ \ldots\ldots}{(1+T_a\ s)\ (1+T_b\ s)\ \ldots\ldots}$$

$$K_a = \lim_{s\to 0} s^2\,G(s)\,H(s) = 0 \qquad \therefore\ e_{ss} = \frac{A}{K_a} = \frac{A}{0} = \infty$$

viii) Consider TYPE 1 system :

$$G(s)\,H(s) = \frac{K\,(1+T_1\,s)\,\,(1+T_2\,s)\,.......}{s(1+T_a\,\,s)\,\,(1+T_b\,s)\,........}$$

$$K_a = \lim_{s \to 0} s^2\,G(s)\,H(s) = 0 \qquad \therefore\; e_{ss} = \frac{A}{K_a} = \infty$$

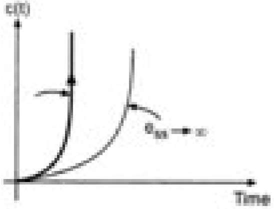
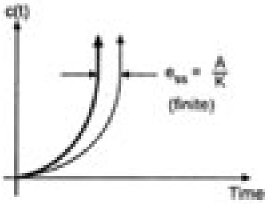


Fig. 6.15 (a) Fig. 6.15 (b)

ix) If TYPE 2 system is used i.e.

$$G(s)\,H(s) = \frac{K\,(1+T_1\,s)\,\,(1+T_2\,s)\,.......}{s^2(1+T_a\,\,s)\,\,(1+T_b\,s)\,........}$$

Then $$K_a = \lim_{s \to 0} s^2\,G(s)\,H(s) = K \qquad \therefore\; e_{ss} = \frac{A}{K_a} = \frac{A}{K} \text{ finite}$$



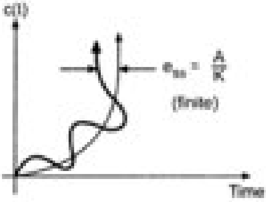


Fig. 6.16 (a) Fig. 6.16 (b)

And for any system of TYPE 3 or more if parabolic input is used error will b negligibly small.

Type of System	Error Coefficients			Error e_{ss} for		
	K_p	K_v	K_a	Step input	Ramp input	Parabolic input
0	K	0	0	$\frac{A}{1+K}$	∞	∞
1	∞	K	0	0	$\frac{A}{K}$	∞
2	∞	∞	K	0	0	$\frac{A}{K}$

Table 6.3

The method is applicable only to stable systems.

Transient Response Analysis

Method to determine total output C(t) :

i) Determine the closed loop transfer function of the system $\frac{C(s)}{R(s)}$.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)\,H(s)}$$

ii) Find the expression for R(s) from the information of reference input R(t) to be applied to the system.

iii) Substitute R(s) in the closed loop T.F. to obtain expression for C(s).

iv) Take Laplace inverse of C(s) by using partial fraction method to obtain total C(t) of the system to the applied input R(t).

As $$C(t) = C_{ss}(t) + C_t(t)$$

The transient output can be studied from the above expression. Transient response may be exponential or oscillatory in nature.

Taking $\lim_{t \to \infty}$ of C(t), the final steady state value C_{ss} of the output also can be obtained.

Analysis of First Order System :

Order : **Order of system is the highest power of 's' in the denominator of a closed loop transfer function.**

Consider a simple system shown in Fig. 6.17 (a)

Find $V_o(t)$ i.e. response if it is excited by unit step input.

$$V_i(t) = 1, \quad t \geq 0$$
$$= 0, \quad t < 0$$

$$\therefore \quad V_i(s) = 1/s$$

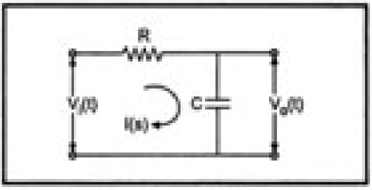


Fig. 6.17 (a)

Now first calculate system T.F.

$$V_i(s) = I(s)\,R + \frac{1}{sC} I(s) \quad \dots (1)$$

$$V_o(s) = \frac{1}{sC} I(s) \quad \dots (2)$$

$$\therefore \quad \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sRC}$$

Substituting $V_i(s) = 1/s$

$$V_o(s) = \frac{1}{s(1+sRC)} = \frac{A'}{s} + \frac{B'}{1+sRC} \qquad A' = 1 \text{ and } B' = -RC$$

$$\therefore \quad V_o(s) = \frac{1}{s} - \frac{RC}{1+sRC}$$

$$= \frac{1}{s} - \frac{1}{s+1/RC}$$

Taking Laplace inverse

$$V_o(t) = 1 - e^{-t/RC} \Rightarrow C_{ss} + C_t(t) \text{ form}$$

So $\quad C_{ss} = 1$ and $C_t(t) = e^{-t/RC}$

t	$V_o(t)$
0	0
RC	0.632
2RC	0.860
3RC	0.950
4RC	0.982
∞	1

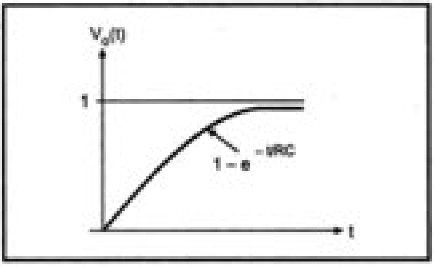


Fig. 6.17 (b)

Ex. 6.1 *For unity feedback system having open loop transfer function* $G(s) = \frac{K(s+2)}{s(s^3+7s^2+12s)}$ *find i) Type of the system ii) Error coefficients iii) Steady state error when input to the system is* $\frac{R}{2}t^2$.

Sol. : Unity feedback system $\therefore$ H(s) = 1

To determine type of the system it is required to bring G(s) H(s) into its time constant form.

$$G(s)H(s) = \frac{K(s+2)}{s(s^3+7s^2+12s)}$$

$$= \frac{2K(1+0.5s)}{s^2(s^2+7s+12)}$$

$$= \frac{2K(1+0.5s)}{s^2(s+4)(s+3)}$$

$$= \frac{2K(1+0.5s)}{s^2 \cdot 4\,(1+0.25\,s)\,3\cdot(1+0.33s)}$$

$$G(s)\,H(s) = \frac{\frac{2K}{12}(1+0.5s)}{s^2\,(1+0.25s)\,(1+0.33s)}$$

Comparing this with,

$$G(s)\,H(s) = \frac{A(1+T_1 s)\,(1+T_2 s)\,.........}{s^j\,(1+T_a\,s)\,(1+T_b s)\,.......}\quad \text{Where } j = \text{Type of the system}$$

$\therefore$ $j = 2$, So system is Type 2 system.

Error coefficients calculation :

$$K_p = \lim_{s \to 0} G(s)\,H(s)$$

$$= \lim_{s \to 0} \frac{\frac{K}{6}(1+0.5s)}{s^2\,(1+0.25s)\,(1+0.33s)} = \infty$$

$$K_v = \lim_{s \to 0} sG(s)\,H(s)$$

$$= \lim_{s \to 0} \frac{\frac{K}{6}(1+0.5s)}{s\,(1+0.25s)\,(1+0.33s)} = \infty$$

$$K_a = \lim_{s \to 0} s^2 G(s)\,H(s)$$

$$= \lim_{s \to 0} \frac{\frac{K}{6}(1+0.5s)}{(1+0.25s)\,(1+0.33s)} = \frac{K}{6}$$

To find error use,

$$e_{ss} = \lim_{s \to 0} \frac{s\,R(s)}{1+G(s)\,H(s)}$$

where $R(s)$ = Laplace Transform of input $r(t)$ to the system

$$r(t) = \frac{R}{2}t^2 \quad , \quad R(s) = \frac{R}{s^3}$$

$$\therefore \quad e_{ss} = \lim_{s \to 0} \frac{s \cdot \frac{R}{s^3}}{1 + \frac{\frac{K}{6}(1+0.5\,s)}{s^2(1+0.25s)(1+0.33s)}}$$

$$= \lim_{s \to 0} \frac{R}{s^2\left[1 + \frac{\frac{K}{6}(1+0.5s)}{s^2(1+0.25s)(1+0.33s)}\right]}$$

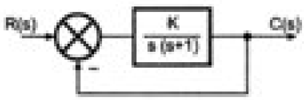
$$= \lim_{s \to 0} \frac{R}{s^2 + \frac{\frac{K}{6}(1+0.5s)}{(1+0.25s)(1+0.33s)}}$$

$$e_{ss} = \frac{6R}{K}$$

Steady state error can also be determined as $e_{ss} = \frac{R}{K_a}$ as system is type 2 system, where R is magnitude of the input.

$$e_{ss} = \frac{R}{\frac{K}{6}} = \frac{6R}{K}$$

Ex. 6.2 *Assuming $r(t) = 0.1\,t$ and it is desired that $e_{ss} \le 0.005$, find the range of values of K for error to be within specified limit for given system.*



Sol. : From the system shown we can write,

$$G(s) = \frac{K}{s(s+1)} \qquad H(s) = 1$$

The input is $r(t) = 0.1\,t$ i.e. ramp of magnitude 0.1. For ramp input K_v controls the error.

$$\therefore \quad K_v = \lim_{s \to 0} sG(s)H(s) = \lim_{s \to 0} \frac{s.K}{s(s+1)} = K$$

$$\therefore \quad e_{ss} = \frac{A}{K_v} = \frac{0.1}{K}$$

Maximum e_{ss} allowed is 0.005

$$\therefore \quad 0.005 = \frac{0.1}{K}$$

$$\therefore \quad K = \frac{0.1}{0.005} = 20$$

For any value of K greater than 20, e_{ss} will be less than 0.005. Hence the range of value of K for $e_{ss} \leq 0.005$ is,

$$20 \leq K < \infty$$

Ex. 6.3 *The block diagram shown in the figure represents a heat treating oven. The set point (desired temperature) is 1000°C. What is steady state temperature?*

Sol. : For the system,

$$G(s) = \frac{20000}{(1+s)\ (1+0.1s)\ (1+0.005s)}$$

H(s) = 1 and input is step of 1000

$$\therefore \quad R(s) = \frac{1000}{s}$$

System is Type 0 system.

For step input,

$$K_p = \lim_{s \to 0} G(s)\,H(s) = \lim_{s \to 0} \frac{20000}{(1+s)\ (1+0.1s)\ (1+0.005s)}$$

$$= 20000$$

$\therefore$ Steady state error,

$$e_{ss} = \frac{A}{1+K_p} \qquad \text{Where A = magnitude of step input}$$

$$\therefore \quad e_{ss} = \frac{1000}{1+20000} = 0.04999$$

$\therefore$ Steady state temperature,

$$\lim_{t \to \infty} C(t) = C_{ss}$$

$$= \text{desired} - e_{ss}$$

$$\therefore \quad C_{ss} = 1000 - 0.04999$$

$$= 999.95^{\circ}\,C.$$

Analysis of Second Order System

Every system has a tendency to oppose the oscillatory behavior of the system which is called **damping.** This damping is measured by a factor or a ratio called **damping ratio** of the system. This damping ratio is denoted by a Greek symbol **(Zeta) ξ.**

The maximum frequency of oscillations under **ξ = 0** condition is called as **Natural frequency of oscillations** of the system and denoted by the symbol **ω_n rad/sec.**
The **C.L.T.F.** For a standard second order system takes the form as:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\,\omega_n\,s + \omega_n^2} \quad \text{... standard 2}^{nd}\text{ order system}$$

Where characteristic equation is, $s^2 + 2\,\xi\,\omega_n\,s + \omega_n^2 = 0$.

e.g. :

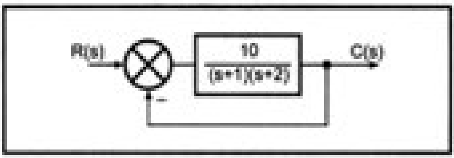


Fig. 6.18

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)} = \frac{\frac{10}{(s+1)(s+2)}}{1+\frac{10}{(s+1)(s+2)}} = \frac{10}{s^2+3s+12}$$

$\therefore \quad \omega_n^2 = 12$ i.e. $\omega_n = \sqrt{12}$ rad / sec

While $2\xi\,\omega_n = 3$ $\therefore\ \xi = \frac{3}{2\sqrt{12}}$

Effect of ξ on Second Order System Performance

Consider input applied to the standard second order system is unit step.

$\therefore \quad R(s) = 1/s$

While $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\,\omega_n\,s + \omega_n^2}$

$$\therefore \quad C(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

Finding the roots of the equation $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$

i.e. $\dfrac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$

i.e. $$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

We can write, $$C(s) = \frac{\omega_n^2}{s\left(s + \xi\omega_n + \omega_n\sqrt{\xi^2-1}\right)\left(s + \xi\omega_n - \omega_n\sqrt{\xi^2-1}\right)}$$

Now nature of these roots is dependent on damping ratio ξ. Consider the following cases,

Case 1 : $1 < \xi < \infty$

The roots are,

$$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$

i.e. **real, unequal and negative, say $-K_1$ and $-K_2$**

$$\therefore \quad C(s) = \frac{\omega_n^2}{s(s+K_1)(s+K_2)} = \frac{A}{s} + \frac{B}{s+K_1} + \frac{C}{s+K_2}$$

Taking Laplace inverse, c(t) will take the following form,

$$c(t) = C_{ss} + Be^{-K_1 t} + Ce^{-K_2 t},$$

where C_{ss} = Steady state output = A

The output is purely exponential. Hence such systems are called **overdamped**

Hence nature of response will be as shown in the Fig. **6.19**

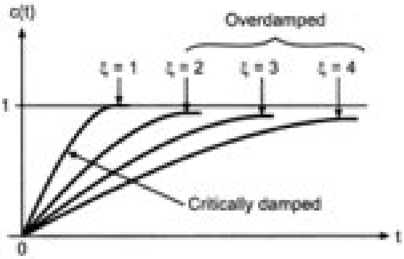


Fig. 6.19 $\xi \geq 1$

Case 2 : $\xi = 1$

The roots are, $s_{1,2} = -\omega_n, -\omega_n$

i.e. **real, equal and negative**.

$$\therefore \quad C(s) = \frac{\omega_n^2}{s(s+\omega_n)(s+\omega_n)} = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{(s+\omega_n)}$$

Taking Laplace Inverse, c(t) will take the following form ...(Refer Chapter-2 for Laplace Inverse).

$$c(t) = C_{ss} + B\,t\,e^{-\omega_n t} + C\,e^{-\omega_n t}$$

where C_{ss} = Steady state output value = A

So $\xi=1$, systems are critically damped and the corresponding response is exponential and as shown in Fig 6.19.

Case 3 : $0 < \xi < 1$

The roots are, $s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

As $\xi < 1$, the term $\sqrt{\xi^2-1}$ is written as $j\sqrt{1-\xi^2}$

Hence roots are **complex conjugates with negative real part**. Note that the real part will be always negative, as ξ or ω_n cannot be negative for a practical system.

$$C(s) = \frac{\omega_n^2}{s(s+\xi\omega_n - j\omega_n\sqrt{1-\xi^2})(s+\xi\omega_n + j\omega_n\sqrt{1-\xi^2})}$$

$$= \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2\xi\omega_n s+\omega_n^2}$$

Taking Laplace Inverse, c(t) will take the following form,

$$c(t) = C_{ss} + K' e^{-\xi\omega_n t}\sin\left(\omega_n\sqrt{1-\xi^2}\,t+\theta\right)$$

Where C_{ss} = Steady state output value = A

$$\theta = \tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{\xi}\right]$$

This response is oscillatory, with oscillating frequency $w_n\sqrt{1-\xi^2}$. This frequency is called **damped frequency of oscillations** w_d, where:

$$\therefore \quad \boxed{\omega_d = \omega_n\sqrt{1-\xi^2} \quad \text{rad/sec}}$$

Such systems are called **Underdamped Systems**. $C(t) = 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$

The nature of the output response for such systems when excited by unit step input will be as shown in the Fig. **6.20**.

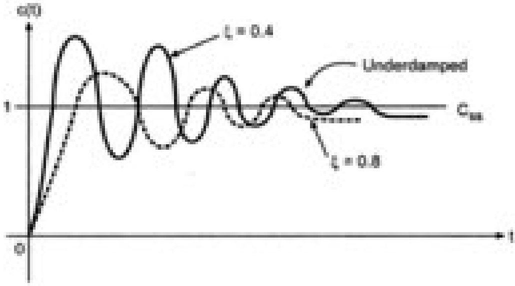


Fig. 6.20 ξ < 1

Case 4 : ξ = 0

The roots are , $s_{1,2} = \pm j\omega_n$

i.e. **complex conjugates with zero real part**. i.e. purely imaginary.

$$\therefore \quad C(s) = \frac{\omega_n^2}{s(s+j\omega_n)(s-j\omega_n)} = \frac{\omega_n^2}{s(s^2+\omega_n^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+\omega_n^2}$$

But instead of finding out partial fractions, the corresponding c(t) can be obtained by substituting ξ = 0 in the expression for c(t) for underdamped condition which is,

$$c(t) = C_{ss} + K' \sin(\omega_n t + \theta) \qquad K' = \text{constant}$$

$$C_{ss} = \text{Steady state output value} = A$$

This response is purely oscillatory. These systems are classified as **undamped systems**. The nature of the response will be as shown in the Fig 6.21.

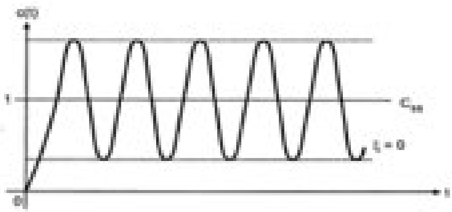


Fig. 6.21 ξ = 0

Summarising all the cases as in the Table 6.4

Sr. No.	Range of ξ	Type of closed loop poles	Nature of response	System classification
1	ξ = 0	Purely imaginary	Oscillations with constant frequency and amplitude	Undamped
2	0 < ξ < 1	Complex conjugates with negative real part	Damped oscillations	Underdamped
3	ξ = 1	Real, equal and negative	Critical and pure exponential	Critically damped
4	1 < ξ < ∞	Real, unequal and negative	Purely exponential slow an sluggish	Overdamped

Table 6.4 Effect of ξ on time response

Transient Response Specifications :

The actual output behaviour according to the expression derived can be shown as shown in Fig. 6.22

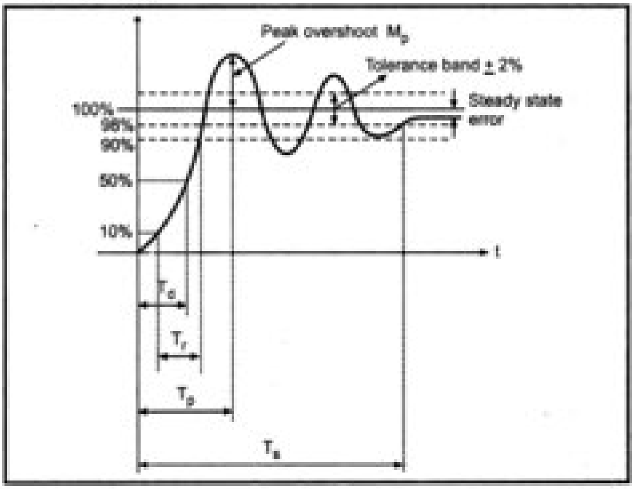


Fig. 6.22

Time Response Specifications :

1) ***Delay Time T_d : It is the time required for the response to reach 50% of the final value in first attempt. It is given by,***

$$T_d = \frac{1 + 0.7\xi}{\omega_n}$$

2) ***Rise Time T_r : It is the time required for the response to rise from 10% to 90% of the final value for overdamped systems and 0 to 100% of the final value for underdamped systems.***

$$T_r = \frac{\pi - \theta}{\omega_d} \text{ sec where } \theta \text{ must be in radians.}$$

3) ***Peak Time T_p : It is the time required for the response to reach its peak value.***

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\xi^2}} \text{ sec}$$

4) ***Peak Overshoot M_p : It is the largest error between reference input and output during the transient period.***

$$\% M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100$$

5) ***Settling Time T_s : This is defined as the time required for the response to decrease and stay within specified percentage of its final value (within tolerance band.)***

$$\text{Time constant of system} = \frac{1}{\xi \omega_n} = T$$

$$T_s = 4 \times \text{Time constant}$$

$$= \frac{4}{\xi \omega_n} \quad \text{..... for a tolerance band of} \pm 2\% \text{ of steady state}$$

1 Time constant 'T' is the time required by the system output to reach 63.2 % of its final value during the first attempt.

Similarly for ± 5% of tolerance band, $T_s = \frac{2.995}{\xi\omega_n} = \frac{3}{\xi\omega_n}$

Ex. 6.4 *A second order system is given by* $\frac{C(s)}{R(s)} = \frac{25}{s^2+6s+25}$. *Find its rise time, peak time, peak overshoot and settling time if subjected to unit step input. Also calculate expression for its output response.*

Sol. : Comparing the T.F. with the standard form $\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

$$\omega_n^2 = 25 \quad \text{and} \quad 2\xi\omega_n = 6$$

$$\omega_n = 5 \qquad \therefore \ \xi = 0.6$$

$$\theta = \tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{\xi}\right] = 0.9272 \text{ radians}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$= 5\sqrt{1-(0.6)^2} = 4 \text{ rad/sec}$$

$$T_r = \frac{\pi-\theta}{\omega_d}$$

$$= \frac{\pi-0.9272}{4} = 0.5535 \text{ sec}$$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{4} = 0.785 \text{ sec}$$

$$\%M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100 = 9.48\ \%$$

$$T_s = \frac{4}{\xi\omega_n} = 1.33 \text{ sec}$$

and

$$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\theta)$$

$$= 1 - \frac{e^{-3t}}{\sqrt{1-(0.6)^2}}\sin(4t+0.9272)$$

$$\therefore \quad C(t) = 1 - 1.5625\, e^{-3t} \sin(4t+0.9272)$$

Ex. 6.5 *A certain control system is described by the differential equation,*

$$\frac{d^2 Y(t)}{dt^2} + 7\frac{dY(t)}{dt} + 12\,Y(t) = 12\,X(t).$$ *Find its output response for unit step input.*

$Y(t)$ = *output*, $X(t)$ = *input*

Sol. : Taking Laplace of the equation neglecting initial conditions.

$$s^2\,Y(s) + 7s\,Y(s) + 12\,Y(s) = 12\,X(s)$$

T.F. is $$\frac{Y(s)}{X(s)} = \frac{12}{s^2 + 7s + 12}$$

Comparing denominator with standard form

$\omega_n = \sqrt{12}$, $2\xi\,\omega_n = 7$ $\therefore \xi = 1.010363$

As $\xi > 1$, system is overdamped, hence the output will not contain any oscillations. Hence **standard expression for C(t) cannot be used.**

Now input is unit step, so $X(s) = 1/s$. Substituting in T.F. we get,

$$Y(s) = \frac{12}{s(s^2 + 7s + 12)}$$ Use partial fraction method

$$= \frac{12}{s(s+3)(s+4)}$$

$$= \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+4}$$

where, $A = 1$, $B = -4$, $C = 3$

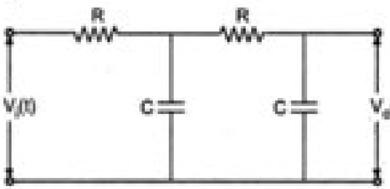
$$Y(s) = \frac{1}{s} - \frac{4}{s+3} + \frac{3}{s+4}$$

Taking Laplace inverse,

$$Y(t) = 1 - 4e^{-3t} + 3e^{-4t}$$

H.W:

For the system shown in figure show that system is always overdamped, independent of the selection of R and C.



Lesson 7

P-I-D Controllers

P-I-D Controller

Objectives:

At the end of this lesson, students will be able to:

1. Reduce the effect of e_{ss} and T_s on time response using PID controller

Introduction to P-I-D Controllers

A controller is a device which when introduced in feedback or forward path of system, controls the steady state and transient response as per the requirement.

Consider such second order system where controller input is error itself and proportional constant is K = 1 as shown in the Fig. 7.1 .

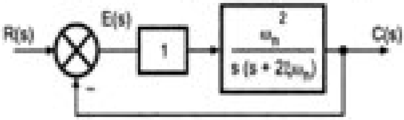


Fig. 7.1

$$G(s)\,H(s) = \frac{\omega_n^2}{s(s + 2\xi\,\omega_n)}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\,\omega_n\,s + \omega_n^2}$$

For this system damping ratio is ξ and natural frequency ω_n.

And for steady state error

$$K_p = \lim_{s \to 0} G(s)\,H(s) = \infty$$

and $$K_v = \lim_{s \to 0} s\,G(s)\,H(s) = \frac{\omega_n}{2\xi}$$

Now if transient response is to be improved, damping ratio must be changed.

In general good time response demands,

i) Less settling time ii) Less overshoot

iii) Less rise time iv) Smallest s.s. error

By increase in K_v i.e. increase in system gain, s.s. error can be reduced but due to high gain settling time and peak overshoot increases. This may lead to instability of system.

So compromise is made to keep steady state error and overshoot within acceptable limits by providing following different types of controllers.

i) PD → Proportional + Derivative Action.

ii) PI → Proportional + Integral Action .

iii) PID → Proportional + Derivative + Integral Action.

iv) Rate feedback controller (Output derivative controller)

PD Type of Controller

A controller in the forward path, which changes the controller output corresponding to proportional plus derivative of error signal is called **PD controller**.

i.e. $$\text{Output of controller} = K\,e(t) + T_d \frac{de(t)}{dt}$$

$$\text{Taking Laplace} = K\,E(s) + sT_d\,E(s) = E(s)\,[\,K + sT_d\,]$$

The T.F. of such controller is $[\,K + sT_d\,]$. This can be realised as shown in the Fig. 7.2

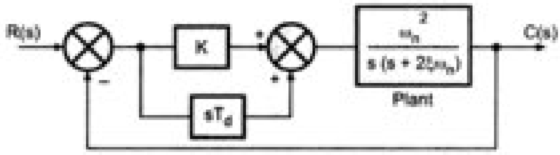


Fig. 7.2

Assuming K = 1, we can write,

$$G(s) = \frac{(1 + sT_d)\,\omega_n^2}{s(s + 2\xi\,\omega_n)}$$

and $$\frac{C(s)}{R(s)} = \frac{(1 + s\,T_d)\,\omega_n^2}{s^2 + s[2\xi\,\omega_n + \omega_n^2\,T_d] + \omega_n^2}$$

Comparing denominator with standard form, ω_n is same as in the previous P type controller.

and $$2\xi'\,\omega_n = 2\xi\,\omega_n + \omega_n^2\,T_d$$

$\therefore$ $$\xi' = \xi + \frac{\omega_n\,T_d}{2}$$

Because of this controller, damping ratio increases by factor $\frac{\omega_n\,T_d}{2}$.

$$K_p = \lim_{s \to 0} G(s)\,H(s) = \infty$$

$$K_v = \lim_{s \to 0} s\,G(s)\,H(s) = \frac{\omega_n}{2\,\xi}$$

As there is no change in coefficients, error also will remain same.

Key Point: *Hence PD controller has following effects on system.*

i) It increases damping ratio.

ii) 'ω_n' for system remains unchanged.

iii) 'TYPE' of the system remains unchanged.

iv) It reduces peak overshoot.

v) It reduces settling time.

vi) Steady state error remains unchanged.

Key Point: *In general P.D. controller improves transient part without affecting steady state.*

PI Type of Controller

A controller in the forward path, which changes the controller output corresponding to the proportional plus integral of the error signal is called **PI controller**.

i.e. $\boxed{\text{Output of controller} = K\, e(t) + K_i \int e(t)\, dt}$

$$\text{Taking Laplace} = K\,E(s) + \frac{K_i}{s}\,E(s) = E(s)\left[K + \frac{K_i}{s}\right]$$

$\therefore$ The T.F. of such controller is $\left[K + \frac{K_i}{s}\right]$ and can be realised as shown in the Fig. 7.3

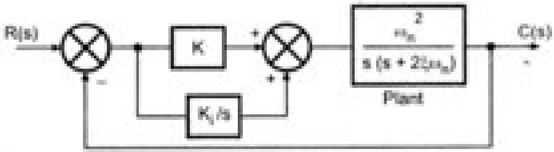


Fig. 7.3

Assuming K = 1, we can write,

$$G(s) = \frac{\left(1 + \frac{K_i}{s}\right)\omega_n^2}{s(s + 2\xi\,\omega_n)}$$

$$= \frac{(K_i + s)\,\omega_n^2}{s^2\,(s + 2\,\xi\,\omega_n)}$$

i.e. system becomes TYPE 2 in nature.

and $$\frac{C(s)}{R(s)} = \frac{(K_i + s)\,\omega_n^2}{s^3 + 2\xi\,\omega_n s^2 + s\,\omega_n^2 + K_i\,\omega_n^2}$$

i.e. it becomes third order.

Now as order increases by one, system relatively becomes less stable as K_i must be designed in such a way that system will remain in stable condition. Second order system is always stable.

Key Point: *Hence transient response gets affected badly if controller is not designed properly.*

While

$$K_p = \lim_{s \to 0} G(s)\,H(s) = \infty, \quad e_{ss} = 0$$

$$K_v = \lim_{s \to 0} s\,G(s)\,H(s) = \infty, \quad e_{ss} = 0$$

Key Point: *Hence as type is increased by one, error becomes zero for ramp type of inputs i state of system gets improved and system becomes more accurate in nature.*

Hence PI controller has following effects :

i) It increases order of the system.

ii) It increases TYPE of the system.

iii) Design of K_i must be proper to maintain stability of system. So it makes system relatively less stable.

iv) Steady state error reduces tremendously for same type of inputs.

Key Point: *In general this controller improves steady state part affecting the transient part.*

PID Type of Controller

As PD improves transient and PI improves steady state, combination of two may be used to improve overall time response of the system. This can be realised as shown in the Fig. 7.4

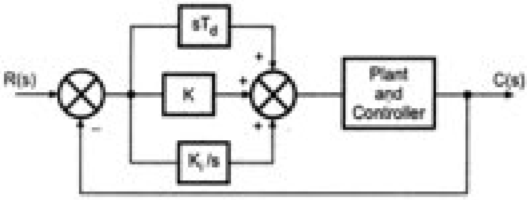


Fig. 7.4

The design of such controller is complicated in practice.

Rate Feedback Controller (Output Derivative Controller)

This is achieved by feeding back the derivative of output signal internally using a tachogenerator and comparing with signal proportional to error as shown. This is called **minor loop feedback compensation.**

$$\text{Output of controller} = K e(t) - K_t \frac{dc(t)}{dt}$$

$\therefore$ Output of the controller $= K\, E(s) - sK_t\, C(s)$ ... taking Laplace

This can be realised as shown in the Fig **7.5** .

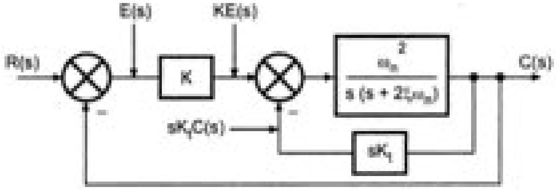


Fig. 7.5

Assuming K = 1, let us study its effect on same system which is considered earlier with $G(s) = \frac{\omega_n^2}{s(s + 2\xi\omega_n)}$.

$$\text{Now new} \qquad G(s) = \frac{\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}}{1 + \dfrac{\omega_n^2}{s(s+2\xi\omega_n)} \cdot s K_t} = \frac{\omega_n^2}{s[s + 2\xi\omega_n + K_t\,\omega_n^2]}$$

$$\text{and} \qquad \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + s[2\xi\omega_n + K_t\,\omega_n^2] + \omega_n^2}$$

$\therefore$ Comparing the denominator it is clear that there is no change in ω_n.

$$\therefore \qquad 2\xi'\omega_n = 2\xi\omega_n + K_t\,\omega_n^2$$

where ξ' = new damping and ξ = old damping

$$\therefore \qquad \xi' = \xi + \frac{K_t\,\omega_n}{2}$$

Key Point: *So there is improvement in damping ratio and hence improvement in transient response.*

$$K_p = \lim_{s \to 0} G(s)\,H(s) = \infty$$

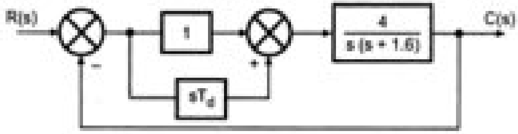
$$K_v = \lim_{s \to 0} s\,G(s)\,H(s)$$

$$= \lim_{s \to 0} \frac{s \cdot \omega_n^2}{s[s + 2\xi\omega_n + K_t\,\omega_n^2]}$$

$$\therefore \quad \boxed{K_v = \frac{\omega_n}{2\xi + K_t\,\omega_n}}$$

Key Point: *As K_t is positive, K_v decreases and as $e_{ss} \propto \frac{1}{K_v}$, s.s. error increases.*

➡ **Example 7.1 :** *The figure shows PD controller used for the system. Determine the value of T_d so that system will be critically damped. Calculate its settling time.*



Solution :

$$G(s) = \frac{(1+sT_d)\,4}{s\,(s+1.6)} \qquad H(s) = 1$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\dfrac{(1+sT_d)\,4}{s\,(s+1.6)}}{1+\dfrac{(1+sT_d)4}{s(s+1.6)}} = \frac{(1+sT_d)\,4}{s^2+1.65+4T_d\,s+4}$$

Comparing denominator with standard form,

$$\omega_n^2 = 4, \quad \omega_n = 2 \quad \text{and} \quad 2\xi\,\omega_n = 1.6 + 4\,T_d$$

$$\therefore \quad \xi = \frac{1.6+4T_d}{4}$$

Now system required is critically damped, i.e. $\xi = 1$

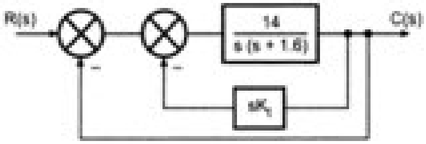
$$\therefore \quad 1 = \frac{1.6+4\,T_d}{4}$$

$$\therefore \quad 4 = 1.6 + 4T_d$$

$$\therefore \quad T_d = 0.6 \quad \text{and settling time} = \frac{4}{\xi\,\omega_n}$$

$$T_s = \frac{4}{2\times 1} = 2\ \text{sec}$$

Example 7.2 : *The system shown in figure uses a rate feedback controller. Determine the tachometer constant K_t so as to obtain the damping ratio as 0.5. Calculate corresponding T_p, M_p, ω_d and T_s.*



Solution :

$$G(s) = \frac{\frac{14}{s(s+1.6)}}{1+\frac{14}{s(s+1.6)}sK_t} = \frac{14}{s^2+1.6s+s\cdot 14K_t} = \frac{14}{s[s+14K_t+1.6]}$$

$$\therefore \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{14}{s(s+14K_t+1.6)}}{1+\frac{14}{s(s+14K_t+1.6)}} = \frac{14}{s^2+s(14K_t+1.6)+14}$$

$$\therefore \omega_n^2 = 14$$

$$\therefore \omega_n = \sqrt{14} = 3.7416 \text{ rad/sec}$$

$$2\xi\omega_n = 14K_t + 1.6$$

$$\therefore \xi = \frac{14K_t+1.6}{2\times\sqrt{14}} = 0.5 \text{ given}$$

$$\therefore K_t = 0.1529$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 3.7416\sqrt{1-(0.5)^2}$$

$$= 3.2403 \text{ rad/sec}$$

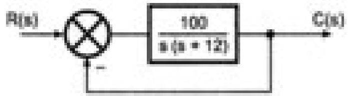
$$T_p = \frac{\pi}{\omega_d} = 0.9695 \text{ sec.}$$

$$\% M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100 = 16.3\ \%$$

$$T_s = \frac{4}{\xi\omega_n} = 2.1381 \text{ sec}$$

Example 7.3 : *For the system shown determine % M_p and T_s when it is excited by unit step input*

If for the same system, PD controller having constant T_d = 1/30 is used in forward path, determine new values of damping ratio, M_p and T_s. Draw respective waveforms



Solution : Without controller,

$$G(s) = \frac{100}{s(s+12)} \quad H(s) = 1$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{100}{s^2 + 12s + 100}$$

$$\therefore \quad \omega_n^2 = 100 \qquad \therefore \omega_n = 10$$

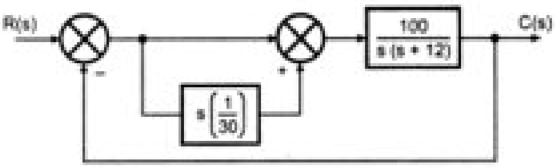
$$2\xi\,\omega_n = 12 \qquad \therefore \quad \xi = 0.6$$

$$\therefore \quad \omega_d = \omega_n \sqrt{1-\xi^2} = 10 \times 0.8 = 8 \text{ rad/sec}$$

$$\therefore \quad \% M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} = \mathbf{9.47\ \%}$$

$$T_s = \frac{4}{\xi\,\omega_n} = \mathbf{0.666\ sec}$$

With controller we get,



$$G(s) = \frac{\left(1 + \frac{s}{30}\right)100}{s(s+12)} = \frac{(s+30) \times 3.33}{s(s+12)}, \quad H(s) = 1$$

$$\frac{C(s)}{R(s)} = \frac{\frac{(s+30)\,3.33}{s(s+12)}}{1 + \frac{(s+30)\,3.33}{s(s+12)}} = \frac{3.33(s+30)}{s^2 + 12s + 3.33s + 100}$$

$$= \frac{3.33\,(s+30)}{s^2 + 15.33s + 100}$$

$$\therefore \quad \omega_n^2 = 100 \qquad \therefore \omega_n = \mathbf{10\ rad/sec}$$

$$2\xi\omega_n = 15.33$$

$$\therefore \quad \xi = \frac{15.33}{2\times 10} = \mathbf{0.7665}$$

$$\therefore \xi \text{ is improved, } \omega_d = \omega_n\sqrt{1-\xi^2} = 10\sqrt{1-(0.7665)^2}$$

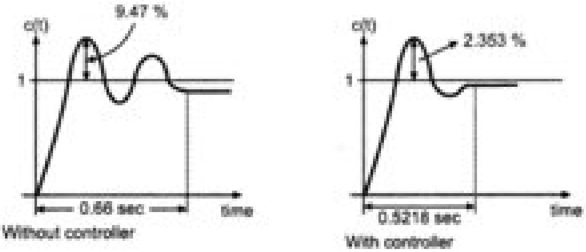
$$= 6.4224 \text{ rad/sec}$$

$$\%\,M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100 = \mathbf{2.353\ \%}$$

Overshoot decreased to 2.3% from 9.47%.

$$T_s = \frac{4}{\xi\omega_n} = \mathbf{0.5218\ sec}$$

Comparison : Following figure shows comparison between system with controller and system without controller.



Lesson 8

Stability of Control System

Stability of Control System

Objectives:

At the end of this lesson, students will be able to:

1. Analyze the stability of control system using Routh-Hurwitz criteria.

Poles of a Transfer Function

Definition : *The values of 's', which make the T.F. infinite after substitution in the denominator of a T.F. are called '**Poles**' of that T.F.*

For example, let the transfer function of a system be,

$$T(s) = \frac{2(s+2)}{s(s+4)}$$

The equation obtained by equating denominator to zero is,

$$s(s+4) = 0$$

$$\therefore \quad s = 0 \quad \text{and} \quad s = -4$$

e.g. For $$T(s) = \frac{2(s+2)}{(s+4)^2\,(s^2+2s+2)\,(s+1)}$$

The poles are the roots of the equation $(s+4)^2\,(s^2+2s+2)(s+1) = 0.$

$\therefore$ Poles are $s = -4, -4, -1 \pm j1, -1$

So T(s) has simple pole at $s = -1,$

Repeated pole at $s = -4,$ (two poles)

Complex conjugate poles at $s = -1 \pm j1$

Poles are indicated by 'X' (cross) in s-plane.

Zeros of a Transfer Function

Definition : *The values of 's' which make the T.F. zero after substituting in the numerator are called '**zeros**' of that T.F.*

e.g. $$T(s) = \frac{2(s+1)^2\,(s+2)\,(s^2+2s+2)}{s^3\,(s+4)\,(s^2+6s+25)}$$

This transfer function has zeros which are roots of the equation,

$$2(s+1)^2\,(s+2)\,(s^2+2s+2) = 0$$

i.e. Simple zero at $s = -2$

Repeated zero at $s = -1$ (twice)

Complex conjugate zeros at $s = -1 \pm j1$.

The zeros are indicated by small circle or zero 'O' in the s-plane.

Pole-Zero Plot

Definition : *Plot obtained by locating all poles and zeros of a T.F. in s-plane is called **pole-zero** plot of a system.*

For a system having T.F. as,

$$\frac{C(s)}{R(s)} = \frac{(s+2)}{s\left[s^2+2s+2\right]\left[s^2+7s+12\right]}$$

The characteristic equation is,

$$s(s^2+2s+2)(s^2+7s+12) = 0 \quad \text{i.e.} \quad s(s^2+2s+2)\ (s+3)(s+4) = 0$$

i.e. System is 5th order and there are 5 poles. Poles are 0, $-1 \pm j$, -3, -4 while zero is located at '-2'.

The corresponding pole-zero plot can be drawn as shown in the Fig. 8.1

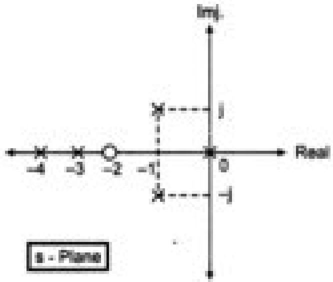


Fig. 8.1

Stability of Control Systems

The analysis of weather the given system can pass through the transient and reach steady state successfully is called **stability analysis of the system**.

Control systems can be:
1. Absolutely stable.
2. Unstable.
3. Conditionally stable.
4. Critically or marginally stable.

Stability of Control Systems

The stability of a linear closed-loop system can be determined from the locations of closed loop poles in the s-plane.

So s-plane can be divided into three distinct zones from stability point of view as shown in the Fig. 8.2

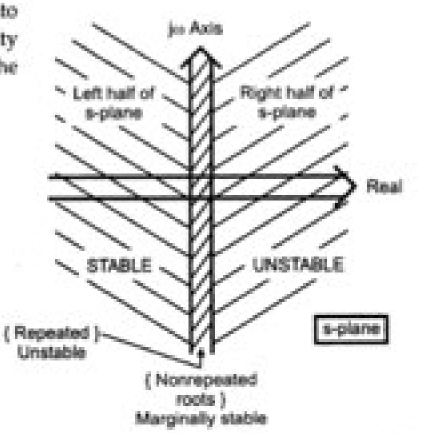
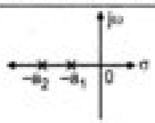
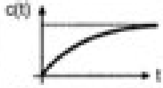
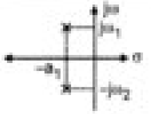
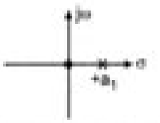
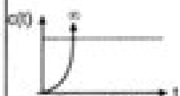


Fig. 8.2 Division of s-plane from stability point of view

Sr. No.	Nature of closed loop poles	Locations of closed loop poles in s-plane	Step response	Stability condition
1.	Real, negative i.e. in L.H.S. of s-plane		 Pure exponential	Absolutely stable
2.	Complex conjugate with negative real part i.e. in L.H.S. of s-plane		 Damped oscillations	Absolutely stable
3.	Real, positive i.e. in R.H.S. of s-plane (Any one closed loop pole in right half irrespective of number of poles in left half of s-plane)		 Exponential but increasing towards ∞	Unstable

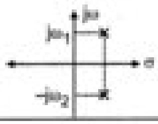
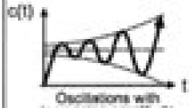
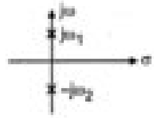
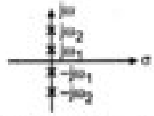
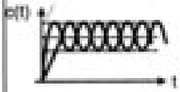
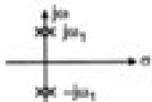
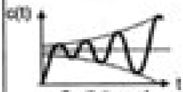
4.	Complex conjugate with positive real part i.e. in R.H.S. of s-plane		 Oscillations with increasing amplitude	Unstable
5.	Nonrepeated pair on imaginary axis without any pole in R.H.S. of s-plane	 or	 Frequency of oscillations = ω_1	Marginally or critically stable
		 two non repeated pairs on imaginary axis.	 Sustained oscillations with two frequency components ω_1 and ω_2	Marginally or critically stable.
6.	Repeated pair on imaginary axis without any pole in R.H.S. of s-plane		 Oscillations of increasing amplitude	Unstable

Table 8.1 Closed loop poles and stability

Routh-Hurwitz Criterion

This represents a method of determining the location of poles of a characteristic equation with respect to the left half and right half of the s-plane without actually solving the equation.

The T.F. of any linear closed loop system can be represented as,

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_n} = \frac{B(s)}{F(s)}$$

where 'a' and 'b' are constants.

To find closed loop poles we equate F(s) = 0. This equation is called **characteristic equation** of the system.

i.e. $$F(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n = 0$$

Thus the roots of the characteristic equation are the closed loop poles of the system which decide the stability of the system.

Hurwitz's Criterion

The necessary and sufficient condition to have all roots of characteristic equation in left half of s-plane is that the sub-determinants D_K , K = 1, 2, ...n obtained from Hurwitz's determinant 'H' must all be positive.

Method of forming Hurwitz determinant :

$$H = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & a_{2n-1} \\ a_0 & a_2 & a_4 & \cdots & a_{2n-2} \\ 0 & a_1 & a_3 & \cdots & a_{2n-3} \\ 0 & a_0 & a_2 & \cdots & a_{2n-4} \\ 0 & 0 & a_1 & \cdots & a_{2n-5} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdot & \cdot & \cdots & a_n \end{vmatrix}$$

$$D_1 = |a_1| \qquad D_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} \qquad D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} \qquad D_K = |H|$$

For the system to be stable, all the above determinants must be positive.

➡ **Example 8.1 :** *Determine the stability of the given characteristic equation by Hurwitz's method.*

$F(s) = s^3 + s^2 + s^1 + 4 = 0$ is characteristic equation.

Solution : $a_0 = 1,\ a_1 = 1,\ a_2 = 1,\ a_3 = 4,\ n = 3$

$$H = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix}$$

$$D_1 = |1| = 1$$

$$D_2 = \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = -3$$

$$D_3 = \begin{vmatrix} 1 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix} = 4 - 16 = -12$$

As D_2 and D_3 are negative, given **system** is **unstable**.

Routh's Stability Criterion

It is also called **Routh's array method** or **Routh-Hurwitz's Method**

Consider the general characteristic equation as,

$$F(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots\ldots + a_n = 0$$

Method of forming an array :

s^n	a_0	a_2	a_4	a_6	......
s^{n-1}	a_1	a_3	a_5	a_7	
s^{n-2}	b_1	b_2	b_3		
s^{n-3}	c_1	c_2	c_3		
$\vdots$	$\vdots$	$\vdots$	$\vdots$		
s^0	a_n				

Coefficients for first two rows are written directly from characteristic equation.

From these two rows next rows can be obtained as follows.

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}, \qquad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}, \qquad b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

From 2^{nd} and 3^{rd} row, 4^{th} row can be obtained as

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}, \qquad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

Routh's Criterion

If there are any sign changes existing then,

a) System is unstable.

b) The number of sign changes equals the number of roots lying in the right half of the s-plane.

Example 8.2 : $s^3 + 6s^2 + 11s + 6 = 0$

Solution : $a_0 = 1, \quad a_1 = 6, \quad a_2 = 11, \quad a_3 = 6, \quad n = 3$

s^3	1	11
s^2	6	6
s^1	$\frac{11 \times 6 - 6}{6} = 10$	0
s^0	6	

As there is no sign change in first column, **system is stable.**

Special Cases of Routh's Criterion

Special Case 1

First element of any of the rows of Routh's array is zero and the same remaining row contains at least one non-zero element.

Effect : The terms in the new row become infinite and Routh's test fails.

e.g. : $s^5 + 2s^4 + 3s^3 + 6s^2 + 2s + 1 = 0$

s^5	1	3	2	
s^4	2	6	1	
s^3	$\boxed{0}$	1.5	0	Special Case 1
s^2	∞	...	...	Routh' array failed

Following two methods are used to remove above said difficulty.

First method : Substitute a small positive number 'ε' in place of a zero occurred as a first element in a row. Complete the array with this number 'ε'. Then examine the sign change by taking $\lim_{\varepsilon \to 0}$. Consider above example.

s^5	1	3	2
s^4	2	6	1
s^3	ε	1.5	0
s^2	$\frac{6\varepsilon - 3}{\varepsilon}$	1	0
s^1	$\frac{\frac{1.5(6\varepsilon - 3)}{\varepsilon} - \varepsilon}{\frac{(6\varepsilon - 3)}{\varepsilon}}$	0	
s^0	1		

To examine sign change.

$$\lim_{\varepsilon \to 0}\left(\frac{6\varepsilon - 3}{\varepsilon}\right) = 6 - \lim_{\varepsilon \to 0}\frac{3}{\varepsilon}$$

$$= 6 - \infty$$

$$= -\infty \text{ sign is negative.}$$

$$\lim_{\varepsilon \to 0}\frac{1.5(6\varepsilon - 3) - \varepsilon^2}{6\varepsilon - 3} = \lim_{\varepsilon \to 0}\frac{9\varepsilon - 4.5 - \varepsilon^2}{6\varepsilon - 3}$$

$$= \frac{0 - 4.5 - 0}{0 - 3}$$

$= +1.5$ sign is positive.

Routh's array is,

s^5	1	3	2
s^4	2	6	1
s^3	$+\epsilon$	1.5	0
s^2	$-\infty$	1	0
s^1	+ 1.5	0	0
s^0	1	0	0

As there are two sign changes, **system is unstable.**

Special Case 2

All the elements of a row in a Routh's array are zero.

Effect : The terms of the next row cannot be determined and Routh's test fails

s^5	a	b	c	
s^4	d	e	f	
s^3	0	0	0	← Row of zeros, Special case 2

This indicates nonavailablility of coefficient in that row.

Procedure to Eliminate this Difficulty

i) Form an equation by using the coefficients of a row which is just above the row of zeros. Such an equation is called an **Auxiliary Equation** denoted as A(s). For above case such an equation is,

$$A(s) = ds^4 + es^2 + f$$

So 'd' is coefficient corresponding to s^4 so first term is ds^4 of A(s).

Next coefficient 'e' is corresponding to alternate power of 's' from 4 i.e. s^2 hence the term es^2 and so on.

ii) Take the derivative of an auxilary equation with respect to 's'.

i.e. $$\frac{dA(s)}{ds} = \underline{4d}\, s^3 + \underline{2e}\, s$$

iii) Replace row of zeros by the coefficients of $\frac{dA(s)}{ds}$

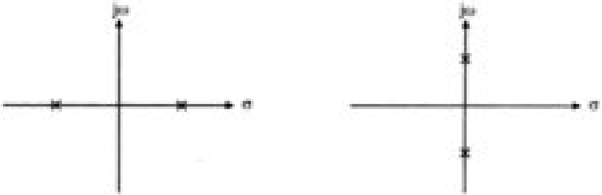
s^5	a	b	c
s^4	d	e	f
s^3	4d	2e	0

iv) Complete the array interms of these new coefficients.

The stability can be predicted from the roots of A(s) = 0 .

The roots of auxiliary equation may be,

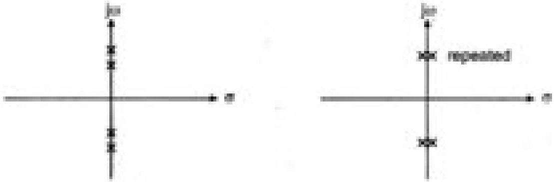
i) A pair of real roots of opposite sign i.e. as shown in the Fig. 8.4 (a).



(a) Fig. 8.4 (b)

ii) A pair of roots located on the imaginary axis as shown in the Fig. 8.4 (b).

iii) The nonrepeated pairs of roots located on the imaginary axis as shown in the Fig. 8.4 (c).



(c) Fig. 8.4 (d)

iv) The repeated pairs of roots located on the imaginary axis as shown in the Fig. 8.4 (d).

Hence total stability can be determined from the roots of $A(s) = 0$, which can be out of four types shown above.

Example 8.3 : *Examine the Stability By Routh's Criterion :*

$$s^4 + 6s^3 + 26s^2 + 56s + 80 = 0$$

Solution :

s^4	1	26	80
s^3	6	56	0
s^2	16.66	80	
s^1	27.21	0	
s^0	80		

As there is no sign change, **system is stable.**

Example 8.4 : $s^6 + 4s^5 + 3s^4 - 16s^2 - 64s - 48 = 0$ *Find the number of roots of this equation with positive real part, zero real part and negative real part.*

Solution :

s^6	1	3	− 16	− 48
s^5	4	0	− 64	0
s^4	3	0	− 48	0
s^3	0	0	0	

$$A(s) = 3s^4 - 48 = 0 \qquad \frac{dA}{ds} = 12s^3$$

s^6	1	3	− 16	− 48
s^5	4	0	− 64	0
s^4	3	0	− 48	0
s^3	12	0	0	0
s^2	[ε] 0	− 48	0	0
s^1	$\frac{576}{\varepsilon}$	0	0	
s^0	− 48			

$$\lim_{\varepsilon \to 0} \frac{576}{\varepsilon} = +\infty$$

∴ One sign change and **system is unstable**. Thus there is one root in R.H.S. of s-plane i.e. with positive real part. Now solve A(s) = 0 for the dominant roots.

$A(s) = 3s^4 - 48 = 0$

Put $s^2 = y$

∴ $3y^2 = 48$ ∴ $y^2 = 16,$ ∴ $y = \pm\sqrt{16} = \pm 4$

∴ $s^2 = +4$ $s^2 = -4$

$s = \pm 2$ $s = \pm 2j$

So $s = \pm 2j$ are the two roots on imaginary axis i.e. with zero real part. Root in R.H.S. indicated by a sign change is $s = +2$ as obtained by solving $A(s) = 0$. Total there are 6 roots as $n = 6$.

Roots with positive real part = 1

Roots with zero real part = 2

Roots with negative real part = 6 − 2 − 1 = 3

Ex. 8.5 *Find range of values of 'K' so that system with the following characteristic equation will be stable.* $F(s) = s\,(s^2 + s + 1)\,(s + 4) + K = 0$

Sol. :

$$F(s) = s\,[s^3 + 5s^2 + 5s + 4] + K = 0$$

$$= s^4 + 5s^3 + 5s^2 + 4s + K = 0$$

s^4	1	5	K
s^3	5	4	0
s^2	4.2	K	0
	0	0	
s^1	$\frac{16.8 - 5K}{4.2}$		
s^0	K		

For system to be stable there should not be sign change in the first column.

$\therefore \quad K > 0 \quad$ from s^0

and $\quad 16.8 - 5K > 0 \quad$ from s^1

$\therefore \quad 16.8 > 5K \qquad \therefore \quad 3.36 > K \qquad \therefore K < 3.36$

$\therefore$ Range of 'K' is $0 < K < 3.36$

$\therefore \quad 16.8 > 5K \qquad \therefore \quad 3.36 > K \qquad \therefore K < 3.36$

$\therefore$ Range of 'K' is $0 < K < 3.36$

Lesson 9

Root Locus Method

Root Locus Method

Objectives:

At the end of this lesson, students will be able to:

1. Sketch the locus of roots in the s-plane as a parameter is varied.
2. Comment on the stability of the system.

Root Locus

It is a graphical method, in which movement of poles in the s-plane is sketched when a particular parameter of system is varied from zero to infinity. For root locus method, gain *(K)* is assumed to be a parameter which is to be varied from zero to infinity.

Rules of Construction of Root Locus

The following rules are applicable in sketching the root locus plot.

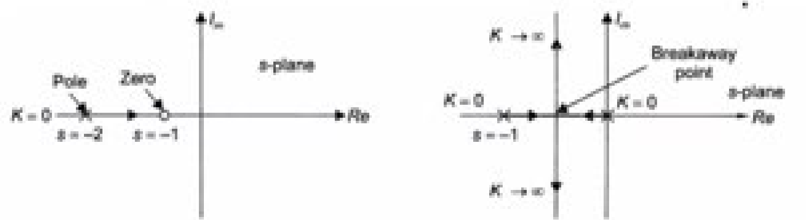
Rule 1: Symmetry of root locus—Any root locus must be symmetrical about the real axis.

Rules 2 and 3: Starting and termination of root loci—Root locus will start from an open-loop pole with gain $K = 0$ and terminate either on an open-loop zero or to infinity with $K = \infty$.

Let us illustrate these rules with an example. Let open-loop transfer functions of control systems are

(1) $G(s)H(s) = \dfrac{K(s+1)}{(s+2)}$

(2) $G(s)H(s) = \dfrac{K}{s(s+1)}$



(a) (b)

Rule 4: Number of root loci—If P is the number or poles and Z is the number of zeros in the transfer function $G(s)H(s)$, the number of root loci N will be as follows:

$$N = P \text{ if } P > Z$$

$$N = P = Z \text{ if } P = Z$$

Rule 5: Root loci on the real axis—The root locus on the real axis will lie in a section of the real axis to the left of an odd number of poles and zeros.

This rule is illustrated through the following examples:

(1) $G(s)H(s) = \dfrac{K(s+2)(s+3)}{s(s+1)}$

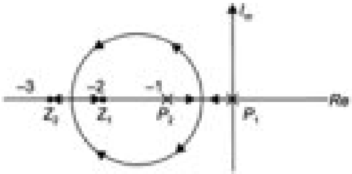


Fig 9.1 Location of root locus on the real axis.

Rule 6: The number of asymptotes and their angles with the real axis—As the value of K is increased to ∞, some branches of root locus from the real axis approach infinity along some asymptotic lines. These asymptotic lines are straight lines originating from the real axis make certain angles with the real axis. The total number of asymptotic lines and the angles they would make are calculated as follows:

Number of asymptotic lines asymptotes

$$= P - Z$$

where P is the number of poles and Z is the number of zeros of the open-loop transfer function, $G(s)H(s)$.

The angle of asymptotes with the real axis is

$$\phi_A = \frac{(2q+1)180^\circ}{P-Z} \quad \text{where } q = 0, 1, 2, \ldots$$

Let us consider,

$$G(s)H(s) = \frac{K}{s(s+2)}$$

Here, the number of poles $P = 2$; they are at $s = 0$ and $s = -2$ and number of zeros $Z = 0$.

Number of asymptotes $= P - Z = 2 - 0 = 2$

Let the angle of two asymptotes be ϕ_A and ϕ'_A respectively. Then,

$$\phi_A = \frac{(2\times 0+1)180^\circ}{2-0} = 90^\circ \quad \text{for } q = 0$$

$$\phi'_A = \frac{(2\times 1+1)180^\circ}{2-0} = 270^\circ \quad \text{for } q = 1$$

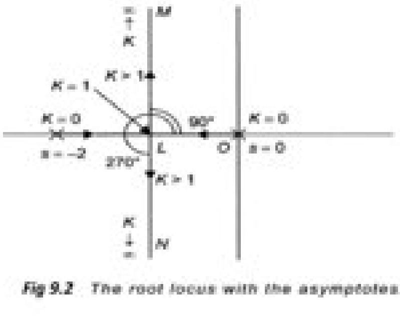


Fig 9.2 The root locus with the asymptotes.

Rule 7: Centroid of the asymptotes—The point of intersection of the asymptotes with the real axis is called the centroid σ_A which is calculated as

$$\sigma_A = \frac{\Sigma \text{ Real parts of poles} - \Sigma \text{ Real parts of zeros}}{P - z}$$

Let us consider the example of Rule 6 where $G(s) = K/s(s + 2)$

$$\sigma_A = \frac{[0 - 2] - [0]}{2 - 0} = -\frac{2}{2} = -1$$

Rule 8: Breakaway points—The root locus breakaway from the real axis where a number of roots are available, normally, where two roots exist.

Consider example 9.4, $G(s)H(s) = \frac{K}{s(s+2)}$

$s = 0$ and $s = -2$

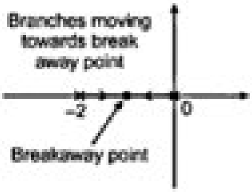


Fig 9.3

Determination of breakaway point :

Steps to determine the co-ordinates of breakaway points are,

Step 1 : Construct the characteristic equation $1 + G(s)H(s) = 0$ of the system.

Step 2 : From this equation, separate the terms involving 'K' and terms involving 's'. Write the value of K in terms of s.

$$K = f(s)$$

Step 3 : Differentiate above equation w.r.t. 's', equate it to zero.

$$\frac{dK}{ds} = 0$$

Step 4 : Roots of the equation $\frac{dK}{ds} = 0$ gives us the breakaway points.

Key Point: *If value of K is positive that breakaway point is valid for the root locus. The breakaway points for which values of K are negative, are invalid for direct root locus but are valid for inverse root locus.*

Example 9.1 : *For* $G(s)H(s) = \dfrac{K}{s(s+1)(s+4)}$, *determine the co-ordinates of valid breakaway points.*

Solution : Characteristic equation $1 + G(s)H(s) = 0$

Step 1 : $1 + \dfrac{K}{s(s+1)(s+4)} = 0$ i.e. $s^3 + 5s^2 + 4s + K = 0$

Step 2 : $K = -s^3 - 5s^2 - 4s$

Step 3 : $\dfrac{dK}{ds} = -3s^2 - 10s - 4 = 0$

Step 4 : $3s^2 + 10s + 4 = 0$

$\therefore$ Breakaway points $= \dfrac{-10 \pm \sqrt{100 - 4 \times 4 \times 3}}{2 \times 3} = -0.46, -2.86$

Substituting in expression for K

For $s = -0.46$, $K = +0.8793$

For $s = -2.86$, $K = -6.064$

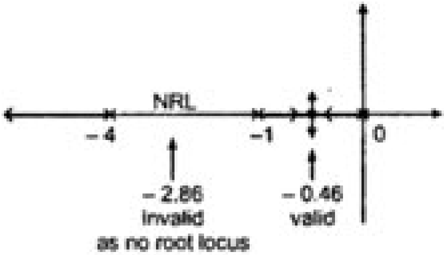


Fig. 9.4

Rule 9: Intersection of root locus with imaginary axis. This can be determined by following procedure.

Step 1 :	Consider characteristic equation $1 + G(s)H(s) = 0$ as obtained in Rule 6.
Step 2 :	Construct Routh's array in terms of "K".
Step 3 :	Determine $K_{marginal}$ i.e. value of K which creates one of the rows of Routh's array as row of zeros, except the row of s^0.
Step 4 :	Construct auxiliary equation $A(s) = 0$ by using coefficients of a row which is just above the row of zeros.
Step 5 :	Roots of auxiliary equation $A(s) = 0$ for $K = K_{mar}$ are nothing but the intersection points of the root locus with imaginary axis.

Consider example 9.1 :

$$G(s)H(s) = \frac{K}{s(s+1)(s+4)}$$

Characteristic equation is given by,

$$1 + G(s)H(s) = 1 + \frac{K}{s(s+1)(s+4)} = 0$$

i.e. $s^3 + 5s^2 + 4s + K = 0$

Routh's array,

s^3	1	4
s^2	5	K
s^1	$\frac{20-K}{5}$	0
s^0	K	

$K_{mar} = 20$ that makes row corresponding to s^1 as row of zeros.

$\therefore \quad A(s) = 5s^2 + K = 0$

$K = K_{mar} = 20$

$5s^2 + 20 = 0$

$s^2 = -4 \quad \therefore \; s = \pm j2$

Key Point: *If K_{mar} is positive, root locus intersects with imaginary axis. But if K_{mar} is negative root locus does not intersect with imaginary axis and lies totally in left half of s-plane.*

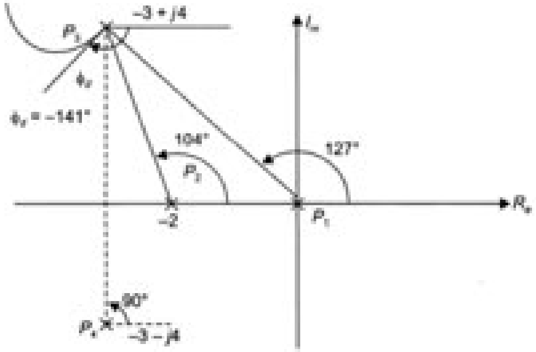
Rule 10: **Angle of departure of the root locus**—The angle of departure of the locus from a complex pole is calculated as

ϕ_d = 180° – sum of angles made by vectors drawn from the other poles to this pole + sum of angles made by vectors drawn from the zeros to this pole.

Let us consider an example. Let

$$G(s)H(s) = \frac{K}{s(s+2)(s^2+6s+25)}$$

$$\phi_d = 180° - (127° + 104° + 90°) + 0 = 180° - 321° = -141°$$



Example 9.2 : *For* $G(s)H(s) = \frac{K(s+2)}{s(s+4)(s^2+2s+2)}$ *, calculate angles of departures at complex conjugate poles.*

Solution : P = 4 , Z = 1

Poles are at s = 0 , −4 , − 1 ± j

Zero at s = −2.

Draw Pole-Zero plot.

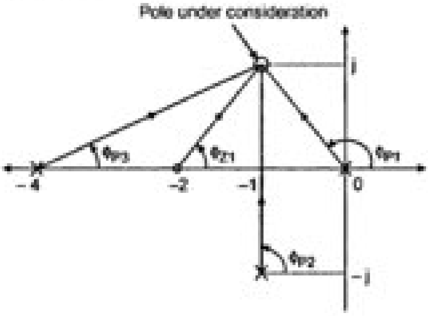


Fig. 9.5

Then , $\sum\phi_P = \phi_{P1} + \phi_{P2} + \phi_{P3}$ while

$$\sum\phi_Z = \phi_{Z1}$$

From geometry of the Fig. 9.17 we can calculate,

$$\phi_{P1} = 135°,\ \phi_{P2} = 90°,\ \phi_{P3} = 18.43°$$

$$\therefore \quad \sum\phi_P = 135° + 90° + 18.43° = 243.43°$$

$$\sum\phi_Z = \phi_{Z1} = 45°$$

$$\therefore \quad \phi = \sum\phi_P - \sum\phi_Z = 243.43° - 45° = 198.43°$$

$$\phi_d = 180° - \phi = 180° - 198.43° = -18.43°$$

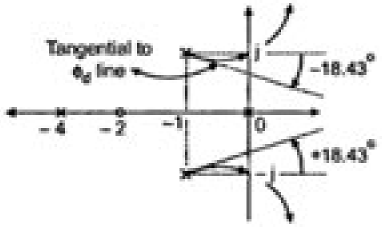


Fig. 9.6

Rule 11: ***Angle of arrival at a complex zero :***

Angle of arrival at a complex zero can be calculated by the same method, which is denoted as ϕ_a. The only change to calculate the angle of arrival is,

$$\phi_a = 180° + \phi$$

where $$\phi = \Sigma\phi_P - \Sigma\phi_Z$$

Obtaining G(s)H(s) from Characteristic Equation

i) Collect the terms of s without K together.
ii) Collect the terms of K together.
iii) Divide the entire equation by polynomial containing the terms of s without K.
This gives the form of equation as $1 + G(s)H(s) = 0$.

For example, if the characteristic equation is given as,

$$s^3 + 7s^2 + 12s + Ks + 10K = 0$$

Then rewrite the equation as,

$$(s^3 + 7s^2 + 12s) + K(s + 10) = 0$$

Then divide entire equation by polynomial in s without K i.e.

$$1 + \frac{K(s+10)}{s^3 + 7s^2 + 12s} = 0$$

$$\text{i.e. } 1 + \frac{K(s+10)}{s(s+3)(s+4)} = 0$$

Comparing this with $1 + G(s)H(s) = 0$ we get,

$$G(s)H(s) = \frac{K(s+10)}{s(s+3)(s+4)}$$

From this root locus can be obtained.

General Steps to Solve the Problem on Root Locus

Step 1 : Get the general information about number of open loop poles, zeros, number of branches etc. from G(s)H(s) .

Step 2 : Draw the pole-zero plot. Identify sections of real axis for the existence of the root locus. And predict minimum number of breakaway points by using general predictions.

Step 3 : Calculate angles of asymptotes.

Step 4 : Determine the centroid. Sketch a separate sketch for step 3 and step 4.

Step 5 : Calculate the breakaway and breakin points. If breakaway points are complex conjugates, then use angle condition to check them for their validity as breakaway points.

Step 6 : Calculate the intersection points of root locus with the imaginary axis.

Step 7 : Calculate the angles of departures or arrivals if applicable.

Step 8 : Combine steps 1 to 7 and draw the final sketch of the root locus.

Step 9 : Predict the stability and performance of the given system by using the root locus.

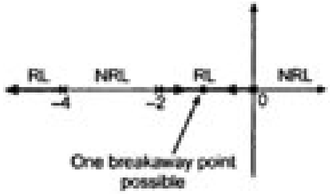
➡ **Example** 9.3 : *For a unity feedback system,* $G(s) = \frac{K}{s(s+4)(s+2)}$. *Sketch the rough nature of the root locus showing all details on it. Comment on the stability of the system.*

(M.U. : June-92)

Solution : Step 1 : General information from $G(s)H(s) = \frac{K}{s(s+2)(s+4)}$

P = 3, Z = 0, number of branches N = P = 3.

Step 2 : Pole-Zero plot and sections of real axis.



Step 3 : Angles of asymptotes.

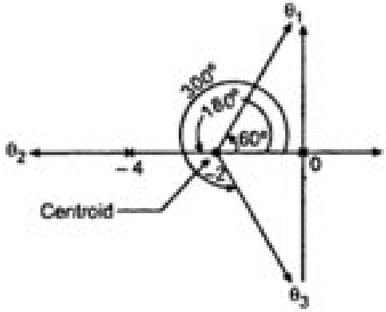
3 branches are approaching to ∞ , 3 asymptotes are required.

$$\theta = \frac{(2q+1)180^\circ}{P-Z}, \qquad q = 0, 1, 2$$

$$\therefore\ \theta_1 = \frac{180^\circ}{3} = 60^\circ,\ \theta_2 = \frac{(2+1)180^\circ}{3} = 180^\circ,\ \theta_3 = \frac{(2\times 2+1)180^\circ}{3} = 300^\circ$$

Step 4 : Centroid

$$\sigma = \frac{\sum \text{R.P. of poles} - \sum \text{R.P. of zeros}}{P-Z} = \frac{0-2-4}{3} = -2$$



Step 5 : To find breakaway point (Refer Rule No. 6). Characteristic equation is

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{s(s+2)(s+4)} = 0$$

$$\therefore \quad s^3 + 6s^2 + 8s + K = 0$$

$$\therefore \quad K = -s^3 - 6s^2 - 8s \qquad \text{... (1)}$$

$$\frac{dK}{ds} = -3s^2 - 12s - 8 = 0$$

i.e. $$3s^2 + 12s + 8 = 0$$

Roots i.e. breakaway points $= \dfrac{-12 \pm \sqrt{144 - 4\times 3\times 8}}{2\times 3} = -0.845\ ,\ -3.15$

For s = − 3.15, K = − 3.079 (Substituting in equation for K)

For s = − 0.845 K = + 3.079

As K is positive **s = − 0.845 is vaild breakaway point.**

Step 6 : Intersection point with imaginary axis.

Characteristic equation

$$s^3 + 6s^2 + 8s + K = 0$$

Routh's array

s^3	1	8
s^2	6	K
s^1	$\frac{48-4}{6}$	0
s^0	K	

$K_{marginal}$ = 48 which makes row of s^1 as row of zeros.

$$A(s) = 6s^2 + K = 0$$

$$K_{mar} = 48$$

$$\therefore\ 6s^2 + 48 = 0$$

$$s^2 = -8$$

$$\therefore \quad s = \pm j\sqrt{8} = \pm j2.828$$

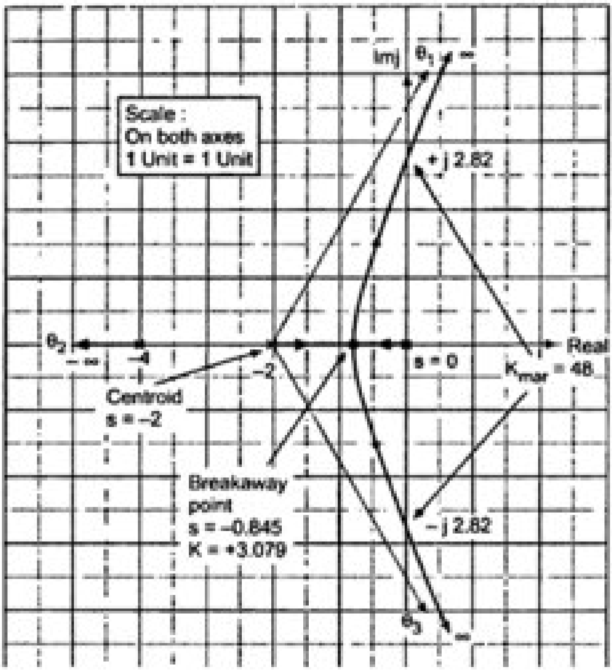
Intersection of root locus with imaginary axis is at ± j 2.828 and corresponding value of $K_{mar} = 48$.

Step 7 : As there are no complex conjugate poles or zeros, no angles of departures or arrivals are required to be calculated.

Step 8 : The complete root locus is as shown below.

Step 9 : Prediction about stability :

For $0 < K < 48$, all the roots are in left half of s-plane hence system is absolutely stable. For $K_{mar} = +48$, a pair of dominant roots on imaginary axis with remaining root in left half. So system is marginally stable oscillating at 2.82 rad/sec. For $48 < K < \infty$, dominant roots are located in right half of s-plane hence system is unstable.



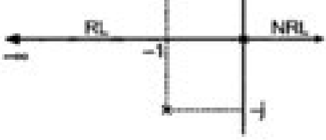
Example 9.4 : *Sketch the root locus for the system having* $G(s)H(s) = \frac{K}{s(s^2+2s+2)}$.

Solution : Step 1 : P = 3, Z = 0, N = P = 3

P − Z = 3 branches approaching to ∞ . Starting points open loop poles.

s = 0, s = − 1 + j , s = −1 − j.

Step 2 : Pole-Zero plot and sections of real axis.



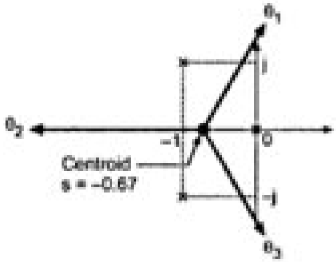
Step 3 : Angles of asymptotes : 3 branches approaching to ∞, 3 asymptotes required

$$\theta = \frac{(2q+1)180^\circ}{P-Z}, \quad q = 0, 1, 2.$$

$$\theta_1 = \frac{180^\circ}{3} = 60^\circ,\ \theta_2 = \frac{(2+1)180^\circ}{3} = 180^\circ,\ \theta_3 = \frac{(2\times 2+1)180^\circ}{3} = 300^\circ$$

Step 4 : Centroid : $\sigma = \frac{\sum \text{R. P. of poles} - \sum \text{R. P. of zeros}}{P-Z} = \frac{0-1-1-0}{3}$

$$= -\frac{2}{3} = -0.67$$



Step 5 : **no breakaway point existing for this system.**

Step 6 : Intersection with imaginary axis.

Characteristic equation : $s^3 + 2s^2 + 2s + K = 0$

Routh's array

s^3	1	2
s^2	2	K
s^1	$\frac{4-K}{2}$	0
s^0	K	

$K_{mar} = +4$ makes row of $s^1 = 0$

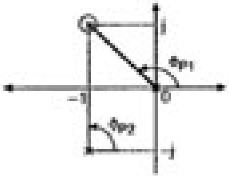
$A(s) = 2s^2 + K = 0$

At $K_{mar} = 4$

$2s^2 + 4 = 0$

$s^2 = -2 \therefore s = \pm j\,1.414$

Step 7 : Angle of departure : As branch is departing at – 1 + j let us calculate angle of departure, at – 1 + j.



$$\phi_{P1} = 135°\ ,\ \phi_{P2} = 90°$$

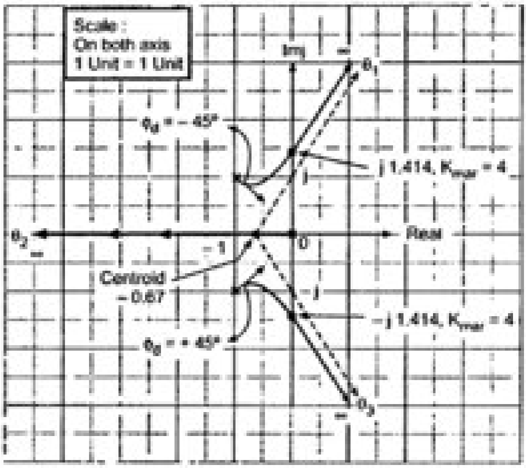
$$\Sigma\phi_P = \phi_{P1} + \phi_{P2} = 225°,\ \Sigma\phi_Z = 0$$

$$\therefore \quad \phi = \Sigma\phi_P - \Sigma\phi_Z = 225°$$

$$\therefore \quad \phi_d = 180° - \phi = 180° - 225° = -45°$$

At $-1-j$, $\phi_d = +45°$

Step 8 : Complete Root Locus is :



Step 9 : Comment on stability :

For 0 < K < 4 all roots are in left half of s-plane. System is absolutely stable.

At K = + 4, dominant roots are on imaginary axis, system is marginally stable, oscillating with 1.414 rad/sec.

At K > 4, dominant roots are in right half of s-plane and hence system becomes unstable in nature.

Example 9.5 : *Sketch the complete root locus for the system having*

$$G(s)H(s) = \frac{K(s+5)}{(s^2+4s+20)}.$$

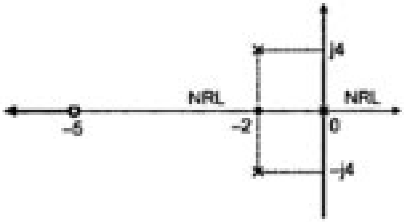
(M.U. : Dec.-2007)

Solution : Step 1 : Number of poles P = 2, Z = 1, N = P

One branch has to terminate at finite zero s = – 5 while P – Z = 1 branch has to terminate at ∞. Starting points of branches are,

$$\frac{-4 \pm \sqrt{16-80}}{2} = -2 \pm j\,4$$

Step 2 : Pole - Zero plot and sections of real axis are as in following figure.



Step 3 : Angles of asymptotes

One branch approaches to ∞ so one asymptote is required.

$$\theta = \frac{(2q+1)\,180^\circ}{P-Z} \quad , q = 0$$

$$\therefore \qquad \theta_1 = 180^\circ$$

Branch approaches to ∞ along + 180° i.e. negative real axis.

Step 4 : Centroid

As there is only one branch approaching to ∞ and one asymptote exists, centroid is not required.

Step 5 : Breakaway point

Characteristic equation : $1 + G(s)H(s) = 0$

$$1 + \frac{K(s+5)}{(s^2+4s+20)} = 0$$

$$\therefore \qquad s^2 + 4s + 20 + Ks + 5K = 0$$

$$\therefore \qquad s^2 + 4s + 20 + K(s+5) = 0$$

$$\therefore \qquad K = \frac{-s^2 - 4s - 20}{(s+5)}$$

Now $$\frac{dK}{ds} = \frac{vu' - uv'}{v^2} = 0$$

$$= (s+5)(-2s-4) - (-s^2 - 4s - 20)(1) = 0$$

$$= -2s^2 - 14s - 20 + s^2 + 4s + 20 = 0$$

i.e. $-s^2 - 10s = 0$

$\therefore \quad -s(s+10) = 0$

s = 0 and s = − 10 are breakaway points. But s = 0 cannot be breakaway point as for s = 0, K = − 4.

For s = − 10 , $K = \dfrac{-100+40-20}{-10+5} = +16$

Hence s = − 10 is valid breakaway point.

Step 6 : Intersection with imaginary axis.

Characteristic equation

$$s^2 + 4s + 20 + Ks + 5K = 0$$

$$s^2 + s(K+4) + (20+5K) = 0$$

Routh's array

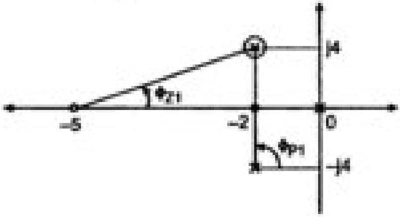
s^2	1	20 + 5 K
s^1	K + 4	0
s^0	20 + 5 K	

$K_{mar} = -4$ makes s^1 row as row of zeros.

But as it is negative, there is no intersection of root locus with imaginary axis.

Step 7 : Angle of departure

Consider − 2 + j4 join remaining pole and zero to it.



$$\phi_{P1} = 90°, \quad \phi_{Z1} = \tan^{-1}\frac{4}{3} = 53.13°$$

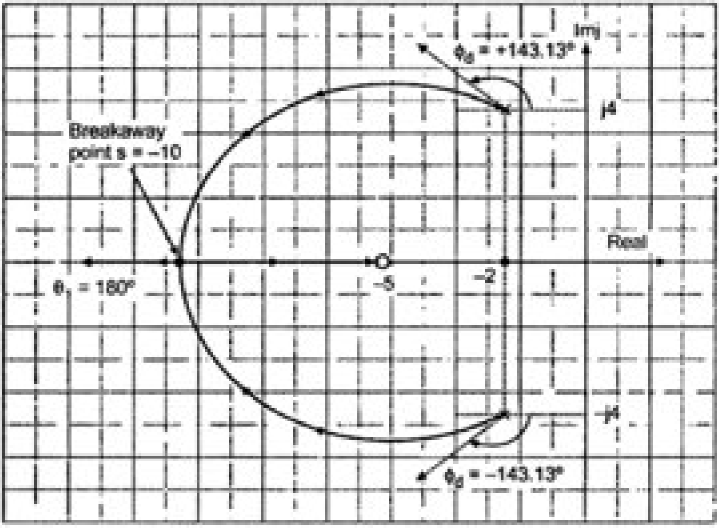
$$\Sigma\phi_P = 90°, \quad \Sigma\phi_Z = 53.13°$$

$$\therefore \quad \phi = \Sigma\phi_P - \Sigma\phi_Z = 36.86°$$

$$\therefore \quad \phi_d = 180° - \phi = +143.13° \quad \text{at} -2+j4 \text{ pole}$$

$$\phi_d = -143.13° \quad \text{at} -2-j4 \text{ pole.}$$

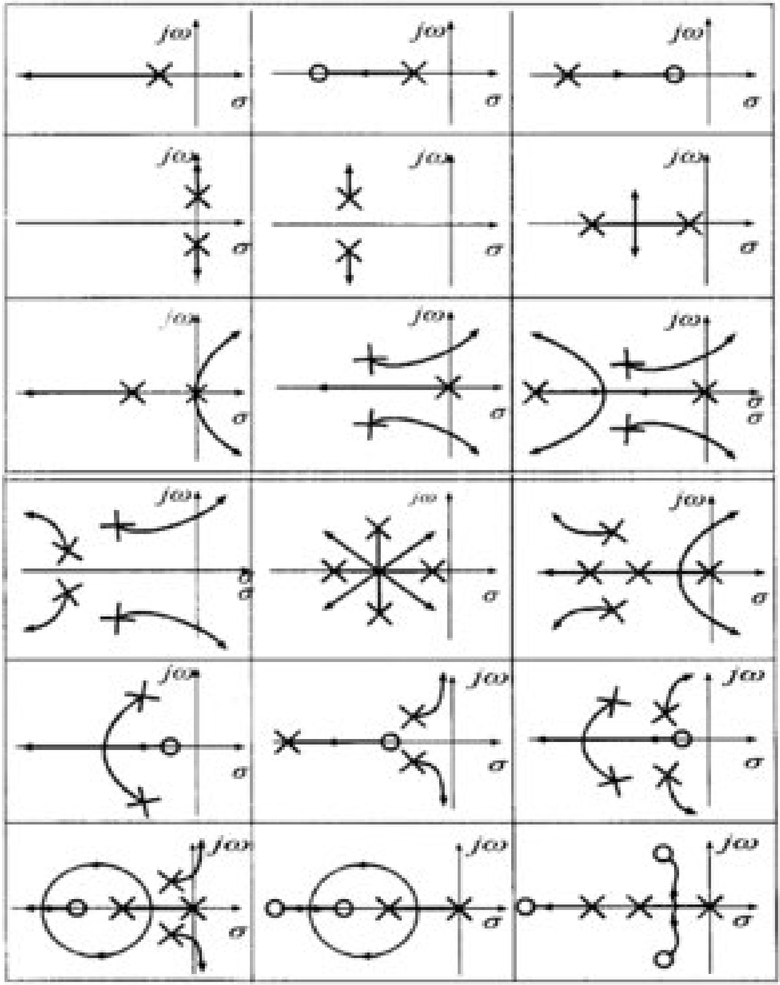
Step 8 : Complete Root Locus is using following figure.



Step 9 : Prediction of stability

For all ranges of K i.e. $0 < K < \infty$, both the roots are always in left half of s-plane. So system is inherently stable.

The Root Locus Method



Lesson 10

Bode Plot Method

Bode Plot Method

Objectives:

At the end of this lesson, students will be able to:

1. Define the frequency response of a system.
2. Use the Bode plot method to find a graph of the frequency response of the system.
3. Comment on the stability of the system.

Basics of Frequency Domain Analysis

Basic of any frequency response is to plot magnitude M and angle ϕ against input frequency `ω'. When `ω' is varied from 0 to ∞ there is wide range of variations in M and ϕ

Such frequency domain transfer function can be obtained by substituting jω for 's' in the transfer function G(s) of the system.

$$G(j\omega) = G(s)|_{s=j\omega} = \text{Frequency domain transfer function}$$

$$G(j\omega) = M\angle\phi$$

M = Magnitude → f(ω)

ϕ = Phase angle → f(ω)

ω = Input frequency

$$|G(j\omega)| = \sqrt{(\text{Real part})^2 + (\text{Imj}\cdot\text{part})^2}$$

$$\phi = \angle G(j\omega) = \tan^{-1}\left[\frac{\text{Imaginary part of } G(j\omega)}{\text{Real part of } G(j\omega)}\right]$$

Example 10.1: find $M\angle\phi$ for the following:

$$G(s) = \frac{10}{s(s+10)}, \quad H(s) = 1$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

i.e.

$$T(s) = \frac{\frac{10}{s(s+10)}}{1+\frac{10}{s(s+10)}}$$

$$\therefore \quad T(s) = \frac{10}{s^2+10s+10}$$

In the frequency domain, replace s by jω

$$\therefore \quad T(j\omega) = \frac{10}{(j\omega)^2+10j\omega+10} = \frac{(10)}{(10-\omega^2)+j10\omega}$$

$$T(j\omega) = M\angle\phi \text{ where}$$

$$M = \frac{|10|}{|(10-\omega^2)+j10\omega|} = \frac{10}{\sqrt{(10-\omega^2)+100\omega^2}}$$

and

$$\phi = -\tan^{-1}\left(\frac{10\omega}{10-\omega^2}\right)$$

Bode Plot

for Bode plot, magnitude in dB and phase angle in degrees are the magnitudes and phase angles of G(jω)H(jω), plotted against Log ω.

> *So in general Bode plot consists of two plots which are,*
>
> *1) Magnitude expressed in logarithmic values against logarithmic values of frequency called Magnitude Plot.*
>
> *2) Phase angle in degrees against logarithmic values of frequency called Phase Angle Plot.*

Magnitude Plot

For Bode Plot $|G(j\omega)| = 20 \log_{10} |G(j\omega)|$ dB.

Such decibels values are to be plotted against $\log_{10}(\omega)$.

So magnitude plot can be shown as in the Fig. 10.1

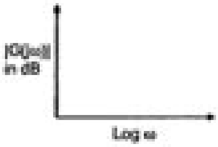


Fig. 10.1 Magnitude plot

The Phase Angle Plot

The phase angle plot can be shown as in the Fig. 10.2

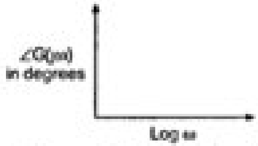


Fig. 10.2 Phase angle plot

Logarithmic Scales (Semilog Papers)

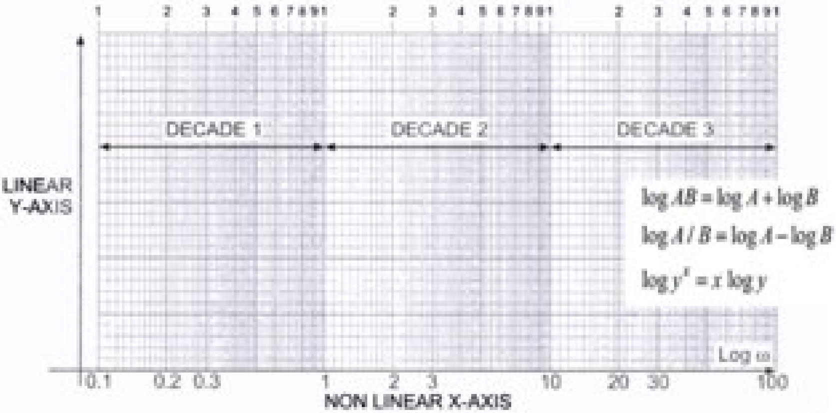


Fig. 10.3 Semilog paper

Standard Form of Open Loop T.F. G(jw)H(jw)

Consider $G(s)H(s) = \dfrac{K' s^Z (s+Z_1)(s+Z_2) \ldots\ldots}{s^P (s+P_1)(s+P_2) \ldots\ldots}$

The standard time constant form can be denoted as,

$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots}{s^P (1+T_a s)(1+T_b s)}$$

K = Resultant system gain P = Type of the system

$T_1, T_2, T_a, T_b, \ldots\ldots$ = Time constants of different poles and zeros.

Frequency domain O.L.T.F.

$$G(j\omega)H(j\omega) = \frac{K(1+T_1 j\omega)(1+T_2 j\omega)\ldots\ldots}{(j\omega)^P (1+T_a j\omega)(1+T_b j\omega)\ldots\ldots}$$

List of such basic factors is ,

1) Resultant system gain K, constant factor.
2) Poles or zeros at the origin. (Integral and Derivative factors) i.e. $(j\omega)^{\pm P}$
3) Simple poles and zeros also called first order factors of the form $(1+j\omega T)^{\pm 1}$
4) Quadratic factors which cannot be factorised into real factors, of the form

$$\left(1+\frac{2\xi}{\omega_n}s+\frac{s^2}{\omega_n^2}\right) = 1+2\xi j\left(\frac{\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2$$

Bode Plots of Standard Factors of G(jw)H(jw)

Factor 1 : System Gain 'K'

Key Point: *This means 'K' shifts the magnitude plot of $|G(j\omega)H(j\omega)|$ by a distance of 20 Log K dB upwards if K > 1 and downwards if K < 1.*

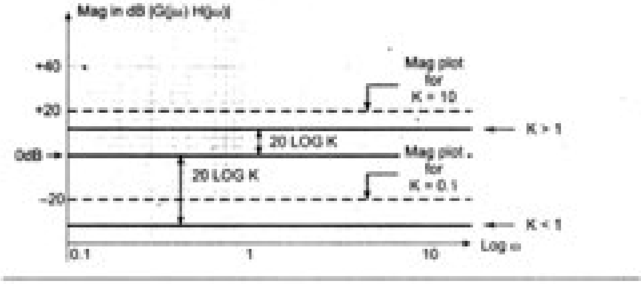


Fig. 10.4 Contribution by K

Phase Angle Plot :

As $G(j\omega)H(j\omega) = K + j0$

Corresponding $\phi = \tan^{-1}\frac{\text{imj part}}{\text{real part}} = \tan^{-1}\frac{0}{K} = 0^\circ$

But if 'K' is negative, it always contributes -180° to the phase angle plot independent of frequency.

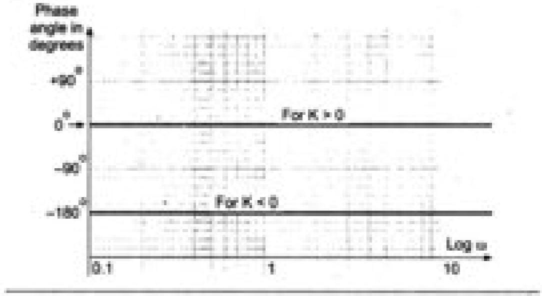


Fig. 10.5

Factor 2 : Poles or Zeros at the Origin $(j\omega)^{\pm P}$

for 'P' number of poles at the origin

$$G(s)H(s) = \frac{1}{s^P}$$

$$G(j\omega)H(j\omega) = \frac{1}{\omega}\cdot\frac{1}{\omega}\ \ldots\ldots\ P \text{ times}$$

$$\therefore \quad |G(j\omega)H(j\omega)| = \frac{1}{\omega}\cdot\frac{1}{\omega}\ \ldots\ldots P \text{ times} = \frac{1}{(\omega)^P}$$

$$|G(j\omega)H(j\omega)| \text{ in dB} = 20 \text{ Log}\frac{1}{(\omega)^P} = 20 \text{ Log}(\omega)^{-P} = -20 \times P \text{ Log } \omega$$

So this is a straight line of slope $-20 \times P$ dB/ decade but again intersecting with 0 dB line at $\omega = 1$.

Key Point: *Therefore magnitude plot for 'P' poles at the origin gives a family of lines passing through intersection of ω = 1 and 0 dB line having slope [− 20 × P] dB/decade as shown in the Fig. 11***10.6**

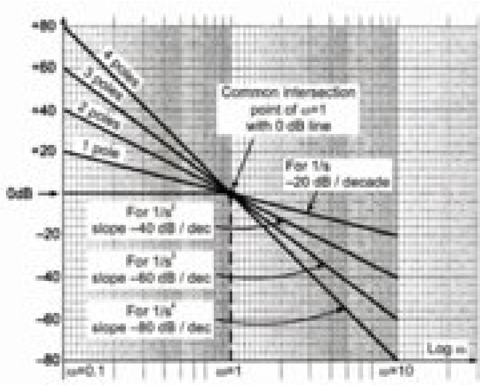


Fig. 10.6 Contribution by poles at origin

for P number of zeros at the origin

$$G(s)H(s) = s^P$$

$$\therefore \quad G(j\omega)H(j\omega) = j\omega \cdot j\omega \cdot j\omega \text{.....P time}$$

$$\therefore \quad |G(j\omega)H(j\omega)| = \omega^P$$

$$\therefore \text{ Magnitude in dB} = 20 \times P \text{ Log } \omega$$

$$\text{i.e.} \quad \text{Slope} = +20 \times P \text{ dB/decade}$$

Key Point : *Each zero at the origin increases the magnitude at a rate of + 20 dB/decade.*

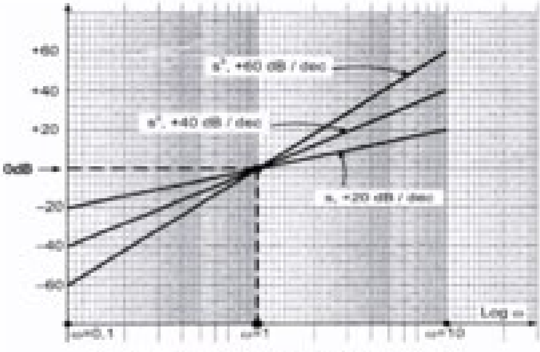


Fig. 10.7 Contribution by zeros at origin

Phase Angle Plot : Consider 1 pole at the origin

$$G(s)H(s) = \frac{1}{s} \quad G(j\omega)H(j\omega) = \frac{1}{j\omega}$$

$$\therefore \quad \angle G(j\omega)H(j\omega) = \angle\frac{1}{j\omega} = \frac{0^\circ}{90^\circ} = -90^\circ$$

For 2 poles at origin,

$$G(s)H(s) = \frac{1}{s^2}$$

$$\therefore \quad G(j\omega)H(j\omega) = \frac{1}{j\omega}\cdot\frac{1}{j\omega}$$

$$\therefore \quad \angle G(j\omega)H(j\omega) = \angle\frac{1}{j\omega}\angle\frac{1}{j\omega} = \frac{0^\circ}{90^\circ \cdot 90^\circ} = -180^\circ$$

Key Point: *In general P number of poles at the origin contribute $-90^\circ \times P$ angle to overall phase angle plot. This contribution is constant irrespective of 'ω'.*

Similarly for a zero at the origin,

$$G(s)H(s) = s \qquad \text{i.e.} \qquad G(j\omega)H(j\omega) = j\omega$$

$$\therefore \quad \angle G(j\omega)H(j\omega) = \angle 0 + j\omega = +\tan^{1}\frac{\omega}{0} = +90^\circ$$

Key Point: *In general 'P' number of zeros at the origin, the total angle contribution is $+90^\circ \times P$, irrespective of value of ω.*

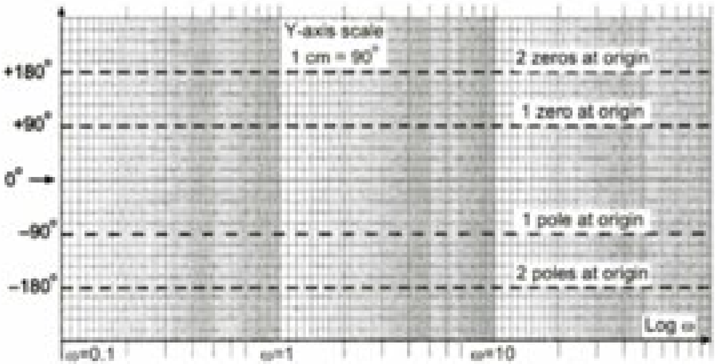


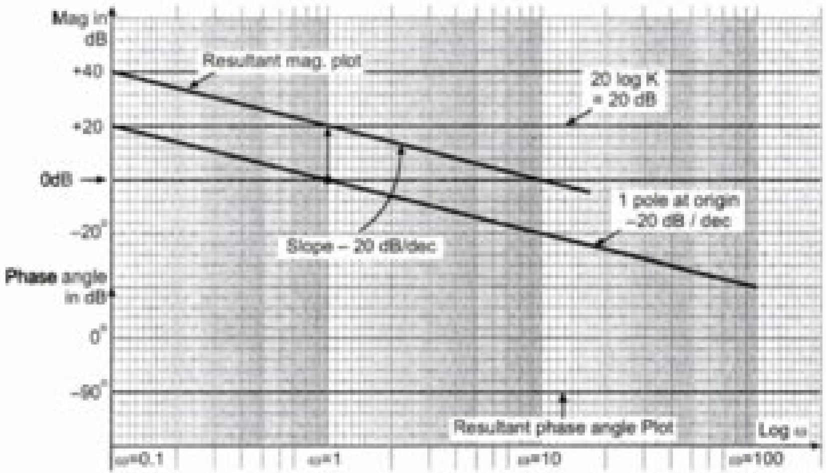
Fig. 10.8 Angle contribution

e.g. Consider $G(s)H(s) = \frac{10}{s}$ so $G(j\omega)H(j\omega) = \frac{10}{j\omega}$

Factor	w_c	Slope	Effect on magnitude plot
K = 10	-	0 dB / dec	Shift up by 20 log 10 = 20 dB/dec
1/ s	-	-20 dB / dec	Slope = 0 – 20 = -20 dB/dec

ω	Contribution by K	By 1 pole at origin	Resultant ϕ_R
0	0°	− 90°	− 90°
10	0°	− 90°	− 90°
40	0°	− 90°	− 90°
1000	0°	− 90°	− 90°
∞	0°	− 90°	− 90°

So phase angle plot is straight line parallel to X-axis as shown with phase angle − 90°.



Factor 3 : Simple Poles or Zeros (First Order Factors)

The factor is represented as $(1+Ts)^{\pm 1}$ i.e. $(1+j\omega T)^{\pm 1}$

Let us start with a simple pole

$$G(s)\,H(s) = (1+Ts)^{-1} = \frac{1}{(1+Ts)}$$

$$G(j\omega)\,H(j\omega) = \frac{1}{1+Tj\omega}$$

$$\therefore \quad |G(j\omega)\,H(j\omega)| = \frac{1}{\sqrt{1+(\omega T)^2}} = \left[\sqrt{1+(\omega T)^2}\right]^{-1}$$

$$\therefore \quad \text{In dB magnitude} = 20 \text{ Log}\left[\sqrt{1+(\omega T)^2}\right]^{-1}$$

$$= -20 \text{ Log}\sqrt{1+\omega^2 T^2} \text{ dB}$$

The approximation is,

i) For low frequency range $\omega << \frac{1}{T}$ i.e. $\omega^2 T^2 << 1$ hence can be neglected.

$\therefore$ Magnitude in dB = $-$ 20 Log 1 = 0 dB.

So for low frequencies it is straight line of 0 dB only. Thus the contribution by such factor can be completely neglected for low frequency range, as it is very small.

ii) For high frequency range $\omega >> \frac{1}{T}$ $\therefore 1 << \omega^2 T^2$

Magnitude in dB = $-$ 20 Log ωT dB

i.e. it is straight line of slope $-$ 20 dB/ decade.

This frequency at which change of slope from 0 dB to $-$ 20 dB/decade occurs is called **Corner Frequency**, denoted by ω_c.

$$\therefore \quad \omega_c = 1/T$$

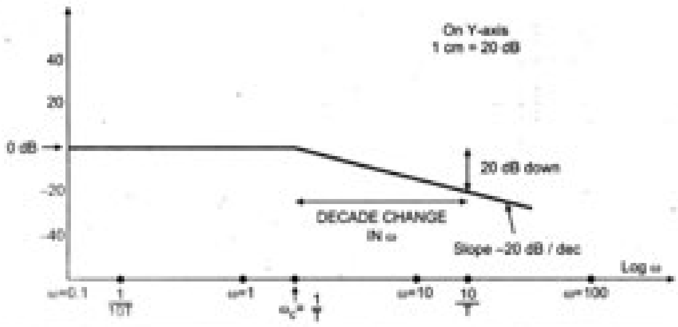


Fig. 10.9

For a simple zero, i.e. first order zero,

$$G(s)H(s) = (1 + Ts) \quad \text{i.e.} \quad G(j\omega)H(j\omega) = (1 + j\omega T)$$

$$\therefore \quad |G(j\omega)H(j\omega)| = \sqrt{1 + (\omega T)^2}$$

$$\therefore \quad \text{Magnitude in dB} = 20 \text{ Log } \sqrt{1 + \omega^2 T^2} \text{ dB}$$

The magnitude plot for simple zero is a straight line of 0 dB upto $\omega_C = 1/T$ and then straight line of slope + 20 dB/decade for all frequencies more than corner frequency.

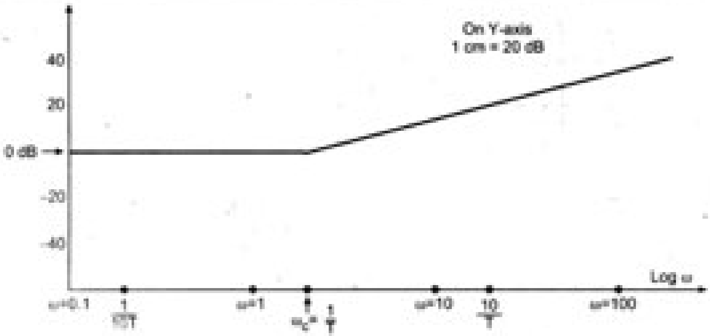


Fig. 10.10

Phase Angle Plot : Consider a simple pole

$$G(s)H(s) = \frac{1}{1 + Ts}$$

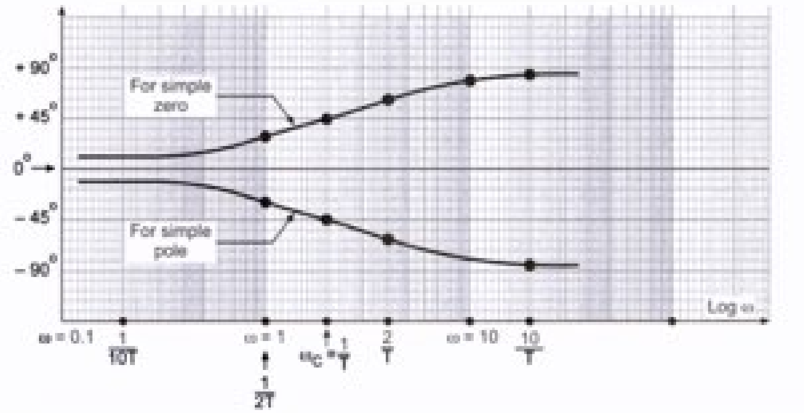
$$G(j\omega)H(j\omega) = \frac{1}{1 + j\omega T} \quad \therefore \angle G(j\omega)H(j\omega) = \frac{0^\circ}{\tan^{-1} \frac{\omega T}{1}} = \tan^{-1} \omega T$$

While for a simple zero,

$$G(s)H(s) = 1 + Ts$$

$$G(j\omega)H(j\omega) = 1 + j\,\omega T \quad \therefore \angle G(j\omega)H(j\omega) = \tan^{-1} \frac{\omega T}{1} = + \tan^{-1} \omega T$$

ω	$\pm \tan^{-1} \omega T$ (+ for zero, – for pole)
$0.1\omega_C = \frac{1}{10T}$	$\pm 5.71°$
$0.5\omega_C = \frac{1}{2T}$	$\pm 26.6°$
$\omega_C = \frac{1}{T}$	$\pm 45°$
$2\omega_C = \frac{2}{T}$	$\pm 63.4°$
$10\omega_C = \frac{10}{T}$	$\pm 84.3°$



Example 10.2 : *Sketch the Bode Plot for the system having*

$$G(s)H(s) = \frac{20}{s(1+0.1\,s)}$$

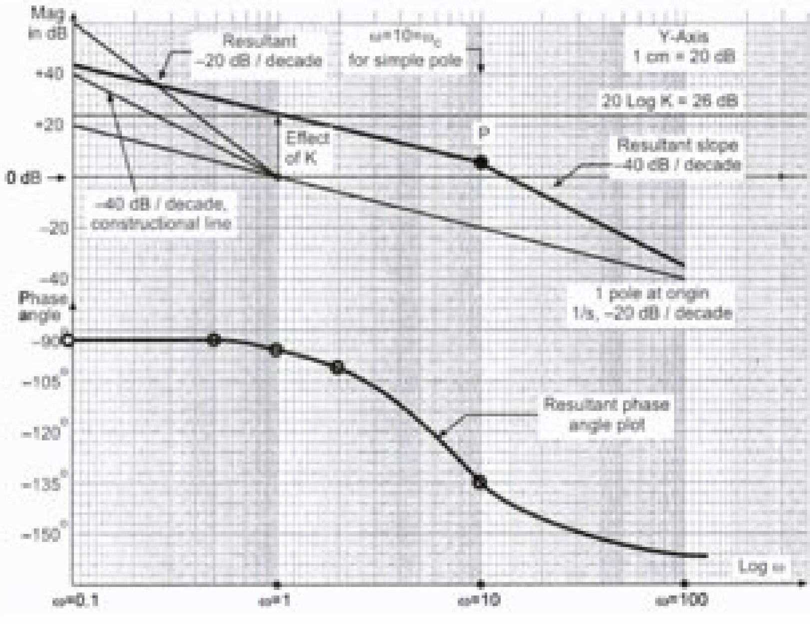
$K = 20 \quad \therefore$ Its magnitude $= 20 \text{ Log } 20 = +\,26$ dB

factor	w_c	slope	Effect on magnitude plot
K=20	-	0 dB/dec	Shift up by 20 log 20 = 26 dB
$\frac{1}{s}$	-	-20 dB/dec	Slope = 0 – 20 = -20 dB/dec
$\frac{1}{1+0.1s}$	10	-20 dB/dec	Slope = -20 – 20 = -40 dB/dec

For the phase angle plot prepare the table of angles as below :

ω in rad/sec	ϕ due to 1 pole at origin	ϕ due to simple pole = – $\tan^{-1}$ 0.1 ω	ϕ_R Resultant
0.1	– 90°	– 0.57°	– 90.57°
0.5	– 90°	– 2.86°	– 92.86°
1	– 90°	– 5.7°	– 95.7°
2	– 90°	– 11.3°	– 101.3°
10	– 90°	– 45°	– 135°
50	– 90°	– 78.79°	– 168°

ϕ due to simple pole = – $\tan^{-1}$ ωT = – $\tan^{-1}$ 0.1 ω



Factor 4 : Quadratic Factors

Consider quadratic pole of the form,

$$G(s)H(s) = \frac{1}{1+\frac{2\xi}{\omega_n}s+\frac{s^2}{\omega_n^2}} \quad \text{expressed in time constant form}$$

$$\therefore \quad G(j\omega)H(j\omega) = \frac{1}{1+2\xi j\left(\frac{\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2}$$

where ω is variable and ω_n is constant for that factor.

$$= \frac{1}{1+2\xi j\left(\frac{\omega}{\omega_n}\right)-\left(\frac{\omega}{\omega_n}\right)^2} \quad \text{as } j^2 = -1$$

$$= \frac{1}{\left\{1-\left[\frac{\omega}{\omega_n}\right]^2\right\}+j\,2\xi\left(\frac{\omega}{\omega_n}\right)}$$

$$\therefore \quad |G(j\omega)H(j\omega)| = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\xi^2\left(\frac{\omega}{\omega_n}\right)^2}}$$

$$\therefore \text{ Magnitude in dB} = 20\,\text{Log}\,\frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\xi^2\left(\frac{\omega}{\omega_n}\right)^2}}$$

$$= 20\,\text{Log}\left(\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\xi^2\left(\frac{\omega}{\omega_n}\right)^2}\right)^{-1}$$

$$\therefore \text{ Magnitude in dB} = -20\,\text{Log}\sqrt{\left(1-\left[\frac{\omega}{\omega_n}\right]^2\right)^2+4\xi^2\left[\frac{\omega}{\omega_n}\right]^2}\ \text{dB}$$

Approximation :

For low frequency, $\omega << \omega_n \quad \therefore \left(\frac{\omega}{\omega_n}\right)^2 << 1$

$\therefore$ Magnitude in dB $= -20\,\text{Log}\,1 = 0$ dB

For high frequency, $\omega > \omega_n$ and $4\xi^2\left(\frac{\omega}{\omega_n}\right)^2 << \left(\frac{\omega}{\omega_n}\right)^4$ as ξ is very low.

$$\therefore \text{ Magnitude in dB} = -20 \text{ Log} \sqrt{\left[\left(\frac{\omega}{\omega_n}\right)^2\right]^2}$$

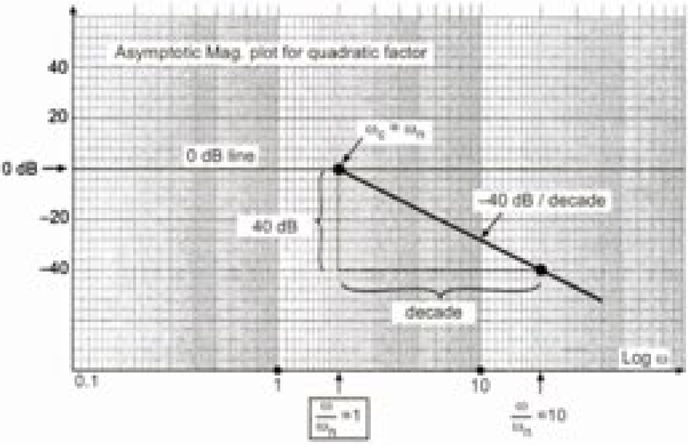
$$= -20 \text{ Log}\left(\frac{\omega}{\omega_n}\right)^2 = -40 \text{ Log } \frac{\omega}{\omega_n}$$

$$= -40 \text{ Log } \omega + 40 \text{ Log } \omega_n$$

This is the equation of straight line of slope – 40 dB/decade.

$\therefore \quad \omega_C = \omega_n$

So ω_n is the corner frequency for such factor.



Let us see the effect of variation of 'ξ' on the magnitude plot.

$$\text{Actual magnitude in dB} = -20 \text{ Log} \sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\xi^2\left(\frac{\omega}{\omega_n}\right)^2}$$

Now at $\omega = \omega_n$ i.e. $\frac{\omega}{\omega_n} = 1$

Actual magnitude in dB = $-20 \text{ Log} \sqrt{4\xi^2}$. Let us prepare a table for various values of

ξ and corresponding error values. See Table 11.10.1

ξ	Accurate magnitude in dB	Approximate magnitude in dB	error for quadratic pole
0.1	+ 13.97	0	+ 13.97 dB up
0.2	+ 7.95	0	+ 7.95 dB up
0.3	+ 4.43	0	+ 4.43 dB up
0.4	+ 1.93	0	+ 1.93 dB up
0.7	− 2.92	0	2.92 dB down
0.9	− 5.10	0	5.1 dB down
1	− 6.02	0	6 dB down

Table 10.1

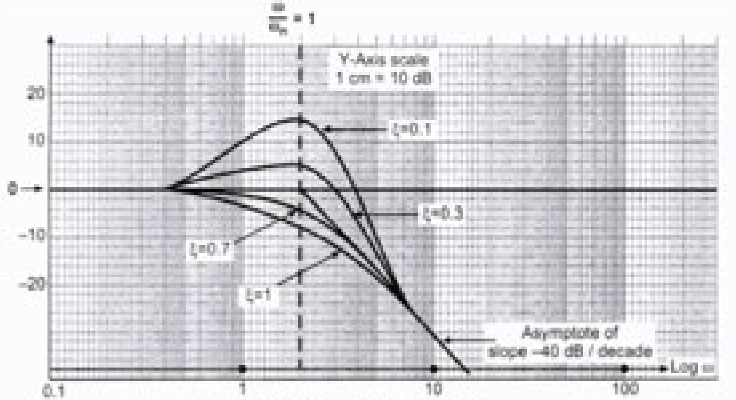


Fig. 10.11 Quadratic pole

Correction = − 20 Log 2ξ dB at $\omega = \omega_n$ of pole

Positive correction upwards and negative correction downwards.

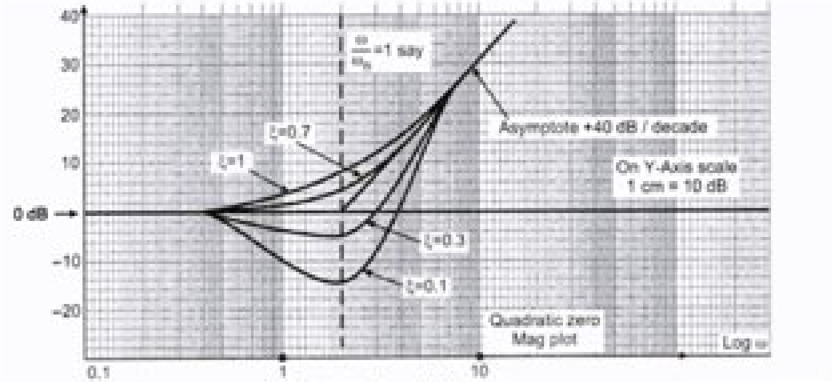


Fig. 10.12 Quadratic zero

Let us see phase angle table :

$$G(j\omega)H(j\omega) = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2 + j\,2\xi\left(\frac{\omega}{\omega_n}\right)} \text{ for a quadratic pole}$$

$$\therefore \quad \angle G(j\omega)H(j\omega) = \frac{0^\circ}{\tan^{-1}\left\{\frac{2\xi(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right\}}$$

$$\angle G(j\omega)H(j\omega) = -\tan^{-1}\left\{\frac{2\xi(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right\}$$

The table for $\xi = 0.3$ is shown below

$$\phi = \tan^{-1}\left\{\frac{2\times 0.3\times \omega/\omega_n}{1-(\omega/\omega_n)^2}\right\}$$

$\frac{\omega}{\omega_n}$	ϕ
0.1	− 3.46°
0.5	− 21.8°
1	− 90°
2	+ 21.8 − 180 = − 158.19°
4	+ 10.09 − 180 = − 170.9°
$\omega > \omega_n$ 10	+ 3.46 − 180 = − 176. 53°
⋮	⋮
∞	− 180

For quadratic zero, sign of the angle should be made positive.

Steps to Sketch the Bode Plot

1) Express given G(s)H(s) into time constant form .
2) Draw a line of 20 Log K dB.
3) Draw a line of appropriate slope representing poles or zeros at the origin, passing through intersection point of $\omega = 1$ and 0 dB.
4) Shift this intersection point on 20 Log K line and draw parallel line to the line drawn in step 3. This is addition of constant K and number of poles or zeros at the origin.
5) Change the slope of this line at various corner frequencies by appropriate value i.e. depending upon which factor is occurring at corner frequency. For a simple pole, slope must be changed by − 20 dB/decade, for a simple zero by + 20 dB/decade etc. **Do not draw these individual lines.** Change the slope of line obtained in step 5 by respective value and draw line with resultant slope. Continue this line till it intersects next corner frequency line. Change the slope and continue. Apply necessary correction for quadratic factor.
6) Prepare the phase angle table and obtain the table of ω and resultant phase angle ϕ_R by actual calculation. Plot these points and draw the smooth curve obtaining the necessary phase angle plot.

Remember that at every corner frequency slope of resultant line must change.

Stability Analysis Using Bode Plot

- Gain Cross Over Frequency (W_{gc}): it is the frequency at which the gain plot intersects the frequency axis (W)
- Phase Cross Over Frequency (W_{pc}): it is the frequency at which the phase plot intersects the frequency axis (W)
- Gain Margin (GM): it is the distance from the W_{pc} and the gain plot on the dB axis
  $GM = 0 - G_0$
- Phase Margin (PM): it is the distance from the W_{gc} and the phase plot on the Ø axis
  $PM = Ø + 180$

Stability Conditions

1- If GM & PM is both positive; the system is stable
2- If GM & PM is both negative; the system is unstable
3- GM & PM never have opposite sign

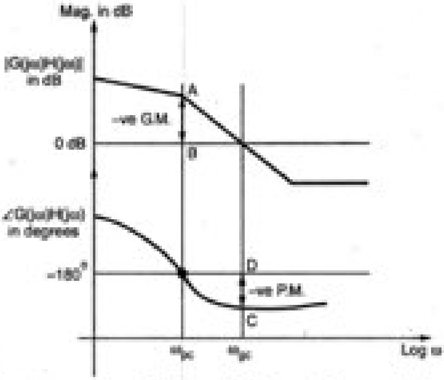


Fig. 10.13 $\omega_{gc} > \omega_{pc}$ G.M. and P.M. negative, unstable system

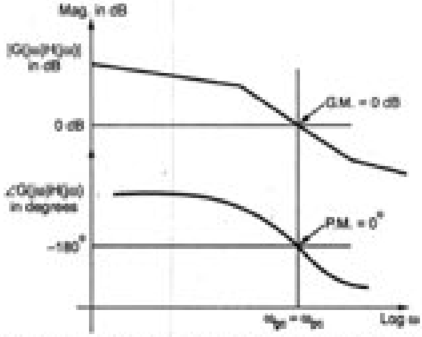


Fig. 10.14 $\omega_{gc} = \omega_{pc}$ G.M. and P.M. zero, marginally stable system

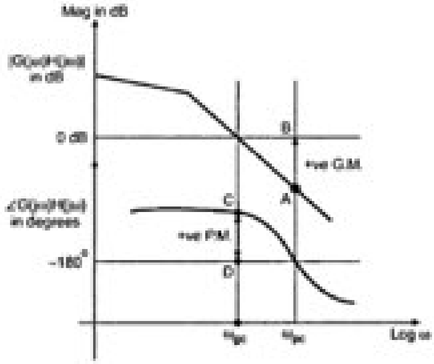


Fig. 10.15 $\omega_{gc} < \omega_{pc}$ G.M. and P.M. positive, stable system

Example 10.3 : *A unity feedback control system has* $G(s) = \frac{80}{s(s+2)(s+20)}$. *Draw the Bode Plot. Determine G.M. P.M.* ω_{gc} *and* ω_{pc}. *Comment on the stability.*

Solution : Step 1 : Arrange G(s)H(s) in time constant form.

$$G(s)H(s) = \frac{80}{s(s+2)(s+20)} \quad , \; H(s) = 1$$

$$= \frac{80}{s(2)\left(1+\frac{s}{2}\right)(20)\left(1+\frac{s}{20}\right)} = \frac{2}{s\left(1+\frac{s}{2}\right)\left(1+\frac{s}{20}\right)}$$

Step 2 : Identify factors :

i) K = 2 ii) 1 pole at origin

iii) Simple pole $\frac{1}{\left(1+\frac{s}{2}\right)}$ with $T_1 = \frac{1}{2}$ $\therefore \omega_{c1} = \frac{1}{T_1} = 2$

iv) Simple pole $\frac{1}{\left(1+\frac{s}{20}\right)}$ with $T_2 = \frac{1}{20}$ $\therefore \omega_{c2} = \frac{1}{T_2} = 20$

Step 3:

factor	w_c	slope	Effect on magnitude plot
K = 2	-	0 dB/dec	Shift up by 20 log 2 = 6 dB
1/s	-	-20 dB/dec	Slope = 0-20 =-20 dB/dec
$\frac{1}{\left(1+\frac{s}{2}\right)}$	2	-20 dB/dec	Slope = -20-20 =-40 dB/dec
$\frac{1}{\left(1+\frac{s}{20}\right)}$	20	-20 dB/dec	Slope = -40-20 =-60 dB/dec

Step 4 : Phase Angle plot : Convert G(s)H(s) to G(jω)H(jω)

$$\therefore \quad G(j\omega)H(j\omega) = \frac{2}{j\omega\left(1+\frac{j\omega}{2}\right)\left(1+\frac{j\omega}{20}\right)}$$

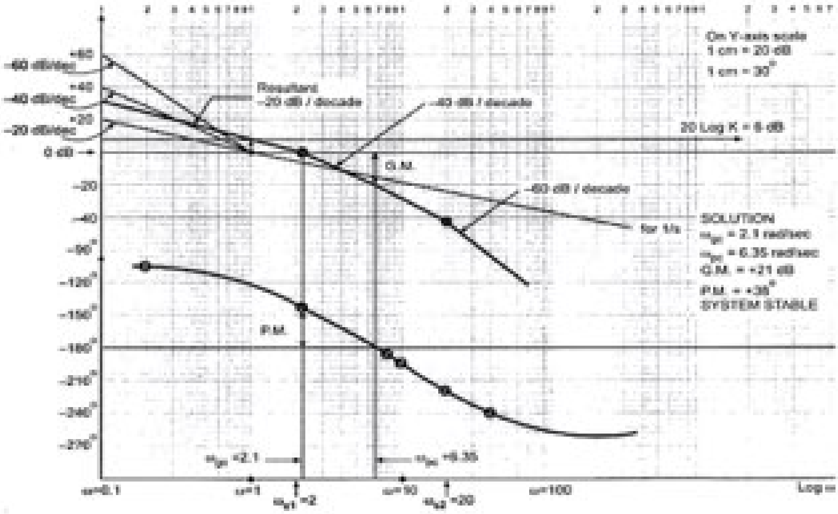
$$\therefore \quad \angle G(j\omega)H(j\omega) = \frac{\angle 2+j\,0}{\angle j\omega \angle 1+\frac{j\omega}{2} \; \angle 1+\frac{j\omega}{20}}$$

$\angle 2 + j0 = 0°$, $\quad \frac{1}{\angle j\omega}$ i.e. 1 pole at origin is – 90° constant

$$\angle \frac{1}{1+j\frac{\omega}{2}} = -\tan^{-1}\frac{\omega}{2} \quad \text{and} \quad \angle \frac{1}{1+j\frac{\omega}{20}} = -\tan^{-1}\frac{\omega}{20}$$

∴ **Phase Angle Table :**

ω	$\frac{1}{j\omega}$	$-\tan^{-1}\frac{\omega}{2}$	$-\tan^{-1}\frac{\omega}{20}$	ϕ_R Resultant
0.2	− 90°	− 5.7°	− 0.57°	− 96.27°
2	− 90°	− 45°	− 5.7°	− 140.7°
8	− 90°	− 75.96°	− 21.8°	− 187.76°
10	− 90°	− 78.69°	− 26.56°	− 195.29°
20	− 90°	− 84.28°	− 45°	− 219.28°
40	− 90°	− 87.13°	− 63.43°	− 240.56°
∞	− 90°	− 90°	− 90°	− 270°



Example 10.4 : *For a particular unity feedback system,* $G(s) = \dfrac{242\,(s+5)}{s\,(s+1)\,(s^2+5s+121)}$

Sketch the Bode Plot. Find ω_{gc} *and* ω_{pc} *, G.M., P.M. Comment on stability.*

Solution : Step 1 : Arrange G(s)H(s) in time constant form.

$$G(s)H(s) = \frac{242 \times 5 \times (1+\frac{s}{5})}{s \times (1+s)(121)(1+\frac{5}{121}s+\frac{s^2}{121})} = \frac{10\,(1+\frac{s}{5})}{s\,(1+s)\,(1+0.041\,s+\frac{s^2}{121})}$$

$\therefore$ **Comparing**

$$\frac{1}{s^2+5s+121} = \frac{1}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$\omega_n^2 = 121 \qquad \therefore \omega_n = 11$$

and $$2\xi\omega_n = 5 \qquad \therefore \xi = \frac{5}{2\times 11} = 0.22$$

$$\text{Correction} = -20\,\text{Log}\sqrt{4\xi^2} = -20\,\text{Log}\,2\xi = -20\,\text{Log}\,(2\times 0.22)$$

$$= +7.13\ \text{dB}$$

Step 2 : Factors :

i) Constant K = 10,

ii) 1 pole at the origin, 1/s

iii) Simple pole , $\frac{1}{1+s}$, $\quad T_1 = 1 \quad \therefore \omega_{C1} = \frac{1}{T_1} = 1$ rad/sec.

iv) Simple zero , $1+\frac{s}{5}$, $\quad T_2 = \frac{1}{5} \quad \therefore \omega_{C2} = \frac{1}{T_2} = 5$ rad/sec.

v) Quadratic pole, $\dfrac{1}{(1+0.0415+\frac{s^2}{121})}$, $\omega_{C3} = \omega_n = 11$ rad/sec.

Step 3 : Magnitude Plot Analysis.

factor	w_c	slope	Effect on magnitude plot
K = 10	-	0 dB/dec	Shift up by 20 log 10 = 20 dB/dec
1/s	-	-20 dB/dec	Slope = 0 -20 =-20 dB/dec
$\frac{1}{1+s}$	$W_{c1} = 1$	-20 dB/dec	Slope = -20 -20 =-40 dB/dec
$1+\frac{s}{5}$	$W_{c2} = 5$	+20 dB/dec	Slope = -40 +20 =-20 dB/dec
$\frac{1}{(1+0.0415+\frac{s^2}{121})}$	$W_{c3} = 11$	-40 dB/dec	Slope = -20 - 40 = -60 dB/dec

Step 4 : Phase Angle Plot.

$$G(j\omega)H(j\omega) = \frac{10\left(1 + j\frac{\omega}{5}\right)}{j\omega(1 + j\omega)\left(1 + 0.041\, j\omega + \frac{(j\omega)^2}{121}\right)} = \frac{10\left(1 + j\frac{\omega}{s}\right)}{j\omega(1 + j\omega)\left(1 + 0.041\, j\omega - \frac{\omega^2}{121}\right)}$$

As $j^2 = -1$

$$\therefore \quad \angle G(j\omega)H(j\omega) = \frac{\angle 10 + j0 \quad \angle(1 + j\frac{\omega}{5})}{\angle j\omega \quad \angle 1 + j\omega \quad \angle 1 + 0.041\, j\omega - \frac{\omega^2}{121}}$$

$$\angle 10 + j0 = 0°, \; 1 + j\frac{\omega}{5} = +\tan^{-1}\frac{\omega}{5}$$

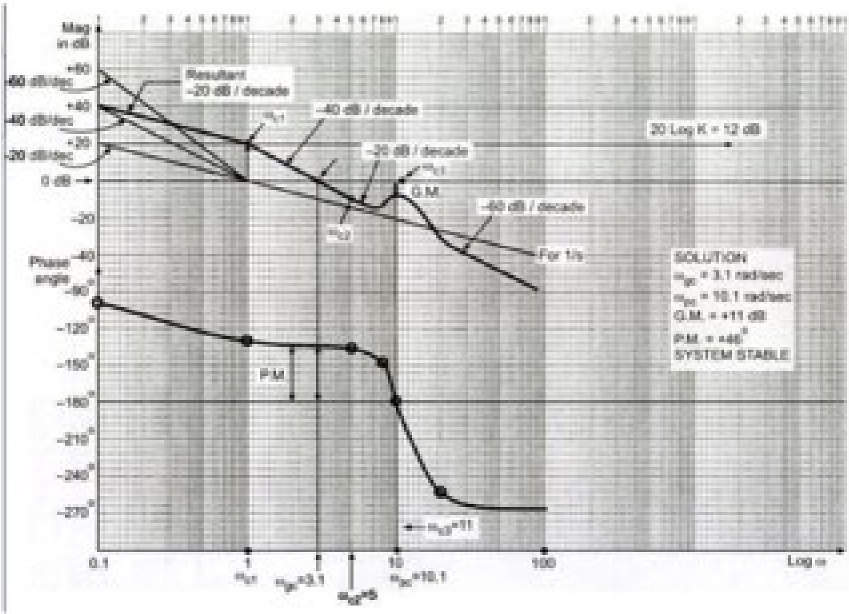
$$\angle\frac{1}{j\omega} = -90° \text{ as 1 pole at origin.}$$

$$\angle\frac{1}{1 + j\omega} = -\tan^{-1}\omega,$$

$$\angle\frac{1}{1 + 0.041\, j\omega - \frac{\omega^2}{121}} = -\tan^{-1}\left\{\frac{0.041\,\omega}{1 - \frac{\omega^2}{121}}\right\}$$

$\therefore$ Phase Angle Table

ω	$\frac{1}{j\omega}$	$+\tan^{-1}\frac{\omega}{5}$	$-\tan^{-1}\omega$	$-\tan^{-1}\left\{\frac{0.041\,\omega}{1-\frac{\omega^2}{121}}\right\}$	ϕ_R
0.1	− 90°	+ 1.14°	− 5.7°	− 0.23°	− 94.7°
1	− 90°	+ 11.3°	− 45°	− 2.36°	− 126.0°
5	− 90°	+ 45°	− 78.6°	− 14.4°	− 138°
8	− 90°	+ 58°	− 82.8°	− 34.8°	− 149.8°
10	− 90°	+ 63.4°	− 84.2°	− 67.0°	− 177.6°
20	− 90°	+ 75.9°	− 87.13°	+ 19.5° − 180° = −160.4°	− 261.63°
∞	− 90°	+ 90°	− 90°	− 180°	− 270°



Example 10.5 : *The transfer function of the system is* $\frac{10(1+0.2s)}{(1+0.5s)}$. *Calculate the phase shift at* $\omega = 10$ *rad/sec.* **(PTU, Jan.-2006)**

Solution : Converting transfer function to the frequency domain,

$$T(j\omega) = \frac{10(1+0.2j\omega)}{(1+0.5j\omega)}$$

$$\therefore \quad \phi = \text{Phase shift} = \angle T(j\omega) = +\tan^{-1}\left(\frac{0.2\omega}{1}\right) - \tan^{-1}\left(\frac{0.5\omega}{1}\right)$$

$$\therefore \quad \phi\big|_{\omega=10} = \tan^{-1}(2) - \tan^{-1}(5) = 63.43° - 78.69° = \mathbf{-15.26°}$$

Transfer Function from Bode Plot

Use the magnitude plot and observe the following points:

1. Starting slope of magnitude plot represents poles or zeros at the origin, if starting slope : -20 dB/ decade there is 1 pole at origin
   -40 dB/ decade there is 2 poles at origin
   0 dB/ decade there is no pole at origin
   +20 dB/ decade there is 1 zero at origin and so on.

2. Observe the shift in magnitude plot at **w =1** which represents **20 log K** from which value of **K** can be determined.
3. Each change in slope indicates the respective factor with corresponding corner frequencies:
   a) If change in slope is -20 dB/dec i.e. -20 to -40 or 0 to -20 and so on then the factor is simple pole.
   b) If change in slope is +20 dB/dec i.e. -40 to -20 or 0 to +20 and so on then the factor is simple zero.

   The respective time constant is reciprocal of the corner frequency $T = \frac{1}{wc}$

The transfer function of the given system can be constructed by taking the product of all of these factors.

➡ **Example 10.6 :** *Determine the transfer function of system whose corner plot is shown below.*

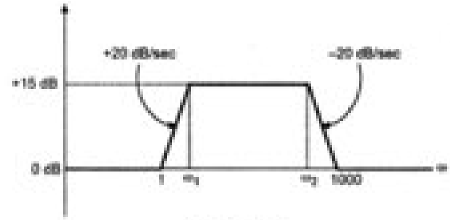


Fig. 10.16

Solution : Starting slope 0 dB so no pole or zero at origin. At each corner frequency slope changes.

Wc	T	Change in slope	Factor
$W_{c1} = 1$	$T_1 = 1$	20 – 0 = 20 dB/dec	Simple zero $(1 + s)$
$W_{c2} = w_1 = 5.623$	$T_2 = 0.177$	0 - 20 = -20 dB/dec	Simple pole $(\frac{1}{1+0.177s})$
$W_{c3} = w_2 = 177.82$	$T_3 = 0.0562$	-20 – 0 = -20 dB/dec	Simple pole $(\frac{1}{1+0.00562s})$
$W_{c4} = 1000$	$T_4 = 0.001$	0 – (-20) = 20 dB/dec	Simple zero $(1 + 0.001s)$

$y = m x + C$ i.e. magnitude in dB = m Log ω + C

$m = +20$ dB/dec

Now at $\omega = 1$, magnitude in dB = 0 dB

Substituting $0 = 20 \text{ Log } 1 + C$

$\therefore$ $C = 0$

$\therefore$ Equation is mag. in dB = 20 Log ω

Now at $\omega = \omega_1$,

Magnitude in dB = 15 dB given

$\therefore$ Substituting $15 = 20 \text{ Log } \omega_1$

$\therefore$ $\omega_1 = 5.623$ rad/sec = ω_{C2}.

Next change in slope is at ω_2.

$\therefore$ $\omega_{C3} = \omega_2$. change is $-20 - 0 = -20$ dB/dec

$\therefore$ Factor is simple pole.

To find ω_2 write equation for that line having slope -20 dB/dec.

Magnitude in dB = m Log ω + C

$\therefore$ Magnitude in dB = -20 Log ω + C.

At $\omega = 1000$, magnitude in dB = 0

Substituting $0 = -20 \text{ Log } 1000 + C$

$\therefore$ $C = +60$ dB

$\therefore$ Equation is, magnitude in dB = -20 Log ω + 60

At ω_2, magnitude in dB = $15 = -20 \text{ Log } \omega_2 + 60$

$\therefore$ $-20 \text{ Log } \omega_2 = -45$

$\therefore$ $\omega_2 = 177.82 = \omega_{C3}$

$\therefore$ The total transfer function is product of all of them.

$$G(s)H(s) = \frac{(1+s)(1+0.001\,s)}{(1+0.177\,s)(1+0.00562\,s)}$$