وزارة التعليم العالي والبحث العلمي
كلية المأمون الجامعة

تحليلات هندسية

المرحلة الثالثة

هندسة الحاسبات

الاستاذة نبراس

Laplace Transform

The main application of Laplace transformation for us will be solving some differential equations, the laplace Trans. converts integral and differential equations into algebric equations. [most diff. eq. with intial values will have a unique solution].

we define the Laplace Transform of a function f, $[0,\infty)$ as:

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt \quad \text{for } s \in \mathbb{C} \text{ (complex num.)}$$

and we use $F(s) = \mathcal{L}\{f(t)\}$

Definition: Given a function $f(t)$, $t \geq 0$, it's Laplace Trans.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \lim_{A\to\infty} \int_0^A e^{-st} f(t)\,dt$$

we say the transform converges if the limit exists and diverges if not

* find the Laplace Trans. using definition

1- $f(t) = 1$ for $t > 0$

Solution: $F(s) = \mathcal{L}(f(t)) = \mathcal{L}(1) = \int_0^\infty e^{-st}(1)\,dt = -\frac{1}{s}\left[e^{-st}\Big|_0^\infty\right]$

$$= -\frac{1}{s}\left[\frac{1}{e^{st}}\Big|_0^\infty\right] = \frac{1}{s}$$

Find the laplace trans. using definition

1) $f(t) = 1$

$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t)\, dt$$

$$= \int_0^{\infty} e^{-st}(1)\, dt = \int_0^{\infty} e^{-st}\, dt \quad \times \frac{-s}{-s}$$

$$= -\frac{1}{s}\int_0^{\infty} -s\, e^{-st}\, dt = \frac{-1}{s}\left(e^{-st}\right)\Big|_0^{\infty} = -\frac{1}{s}\, \frac{1}{e^{st}}\Big|_0^{\infty}$$

$$= -\frac{1}{s}\left(\frac{1}{e^{\infty}} - \frac{1}{e^{0}}\right) = -\frac{1}{s}\left(\cancel{\frac{1}{\infty}}^{\,0} - \frac{1}{1}\right) = \frac{1}{s}$$

$$\therefore \mathcal{L}(1) = \frac{1}{s}$$

$$\mathcal{L}\left(\frac{2}{3}\right) = \frac{2}{3s}$$

$$\mathcal{L}(\sqrt{3}) = \frac{\sqrt{3}}{s}$$

2- $f(t) = 3$

Solution:- $F(s) = \mathcal{L}(f(t)) = \mathcal{L}(3) = \int_0^\infty e^{-st}(3)\,dt = = \frac{3}{s}$

In general If $\boxed{f(t) = a \Rightarrow F(s) = \frac{a}{s}}$

3- $f(t) = u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$ unit step function

Solu: $\mathcal{L}\{u(t)\} = \underbrace{\cancel{\int e^{-st}(0)\,dt}}_{0} + \int_0^\infty e^{-st}(1)\,dt$

$= \int_0^\infty e^{-st}\,dt \Rightarrow [ex(1)]$

4- $f(t) = t$

Solu: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st}\, t\, dt$ using (ud

$\int u\,dv = uv - \int v\,du$

$\boxed{\begin{array}{l} u = t \Rightarrow du = dt \\ dv = e^{-st}dt \Rightarrow v = -\frac{1}{s} e^{-st} \end{array}}$

$\int_0^\infty e^{-st}\, t\, dt = \underbrace{\frac{-t}{s} e^{-st} \Big|_0^\infty}_{0} - \int_0^\infty -\frac{1}{s} e^{-st}\,dt$

$= -\frac{1}{s^2}\left(\frac{1}{e^{st}} \Big|_0^\infty \right) = \frac{1}{s^2}$

In general $\boxed{\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}}$ $n \geq 1$

$f(t) = t^3$

Solu: $\mathcal{L}(t^3) = \frac{3!}{s^4} = \frac{6}{s^4}$

6- The laplace trans. of (Sinat) and (cos at)

Solu: we can find $\mathcal{L}(\sin at)$ and $\mathcal{L}(\cos at)$ using $(u\,dv)$ twice

or by the Euler's formula ; $e^{iat} = \cos at + i \sin at$

$\Rightarrow \mathcal{L}[e^{iat}] = \int_0^\infty e^{-st} e^{iat} dt = \int_0^\infty e^{-(s-ia)t} dt = \frac{1}{s-ia}$

$\therefore \mathcal{L}[e^{iat}] = \frac{1}{s-ia} \cdot \frac{s+ia}{s+ia} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i \frac{a}{s^2+a^2}$

Comparing the real and the imaginary parts we get

$\boxed{\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}}$, $\boxed{\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}}$ $s > 0$

$\therefore \mathcal{L}\{\cos 5t\} = \frac{s}{s^2+25}$ and $\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$

H.w find $\mathcal{L}\{\cosh at\}$ and $\mathcal{L}\{\sinh at\}$

Ex

1) $\mathcal{L}(t^2) = \frac{2!}{s^3} = \frac{2}{s^3}$

2) $\mathcal{L}(t^5) = \frac{5!}{s^6} = \frac{5*4*3*2*1}{s^6} = \frac{120}{s^6}$

3) $\mathcal{L}(\cos 2t) = \frac{s}{s^2+4}$

4) $\mathcal{L}(\cos \sqrt{3}\, t) = \frac{s}{s^2+3}$

5) $\mathcal{L}(\sin \frac{2}{3} t) = \frac{\frac{2}{3}}{s^2+\frac{4}{9}}$

6) $\mathcal{L}(\cosh 5t) = \frac{s}{s^2-25}$

7) $\mathcal{L}(\sinh 7t) = \frac{7}{s^2-49}$

Find $\mathcal{L}(\sin at)$ and $\mathcal{L}(\cos at)$

using Euler's formula ! $e^{iat} = \cos at + i \sin at$

$\therefore \mathcal{L}(e^{iat}) = \int_0^\infty e^{-st} e^{iat}\, dt = \int_0^\infty e^{-st+iat}\, dt = \int_0^\infty e^{-(s-ia)t}\, dt$

$\frac{-1}{(s-ia)} e^{-(s-ia)t} \Big|_0^\infty = \frac{1}{s-ia}$

$\therefore \mathcal{L}(e^{iat}) = \frac{1}{s-ia} * \frac{s+ia}{s+ia} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i \frac{a}{s^2+a^2}$

$\therefore \mathcal{L} \cos at = \frac{s}{s^2+a^2}$ / $\mathcal{L} \sin at = \frac{a}{s^2+a^2}$

$$\boxed{\mathcal{L}(e^{\pm at}) = \frac{1}{s \mp a}}$$

Ex

1) $\mathcal{L}(e^{2t}) = \dfrac{1}{s-2}$

2) $\mathcal{L}(e^{-3t}) = \dfrac{1}{s+3}$

3) $\mathcal{L}(e^{t}) = \dfrac{1}{s-1}$

4) $\mathcal{L}(e^{\frac{-2t}{3}}) = \dfrac{1}{s+\frac{2}{3}}$

5) $\mathcal{L}(9e^{\pi t}) = 9\mathcal{L}(e^{\pi t}) = 9\left(\dfrac{1}{s-\pi}\right) = \dfrac{9}{s-\pi}$

6) $\mathcal{L}\,\frac{3}{2}t^3 = \frac{3}{2}\mathcal{L}\,t^3 = \frac{3}{2} \cdot \dfrac{3!}{s^4} = \frac{3}{2}\,\dfrac{6}{s^4} = \dfrac{9}{s^4}$

7) $\mathcal{L}\left(3\cos t - \dfrac{\sinh 4t}{2} + 5e^{-\sqrt{3}t}\right)$

$= 3\,\dfrac{s}{s^2+1} - \dfrac{1}{2}\,\dfrac{4}{s^2-16} + 5\,\dfrac{1}{s+\sqrt{3}} = \dfrac{3s}{s^2+1} - \dfrac{2}{s^2-16} + \dfrac{5}{s+\sqrt{3}}$

Elementary Properties of Laplace Tra

Suppose $a, b \in \mathbb{C}$ and f, g functions for whic laplace Transform exists ~~then~~:-

1- $\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$ [Linearity]

2- $\mathcal{L}\{e^{\pm at} f(t)\} = F(s \mp a)$ [first shifting property]

3- suppose f and all its derivatives up to and including orde "n" are continous on $[0, \infty)$ and each derivative have in Laplace transformation, then

$$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$$

4- If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{\int_0^t f(u)\,du\} = \frac{F(s)}{s}$

5- $\mathcal{L}\{f(at)\} = \frac{1}{a} F(\frac{s}{a})$ [Time Scaling Property]

6- First derivative: $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

Second derivative $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

and higher order derivatives:-

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

$$\boxed{\mathcal{L}\left(e^{\pm at} f_{(t)}\right) = F(s \mp a)}$$

① أخذ $\mathcal{L} f_{(t)} \Rightarrow F_{(s)}$

② نبدل كل S ب $(S \pm a)$

عكس الاشارة

Ex

1) $\mathcal{L}\, e^{-t} \cos\sqrt{2}\,t =$

① $\mathcal{L} \cos\sqrt{2}\,t = \dfrac{s}{s^2 + 2}$

$\therefore \mathcal{L}\, e^{-t} \cos\sqrt{2}\,t = \dfrac{s+1}{(s+1)^2 + 2}$

2) $\mathcal{L}\left(e^{4t}\, t\right)$

$\because \mathcal{L}\, t = \dfrac{1}{s^2} \Rightarrow \mathcal{L}\, e^{4t}\, t = \dfrac{1}{(s-4)^2}$

$$\boxed{\mathcal{L}\left[t^n f_{(t)}\right] = (-1)^n F^{(n)}_{(s)}}$$

① نأخذ $\mathcal{L} f_{(t)}$

② نشتق $F_{(s)}$ بعدد اس الـ (t) في القانون [بالعدد n]

③ نضرب الناتج النهائي ب $(-1)^n$

$\mathcal{L}\, t^2 \sin 3t$:-

① $\mathcal{L} \sin 3t = \dfrac{3}{s^2+9} = F(s)$

② نشتق مرتين $\Rightarrow F'_{(s)} = \dfrac{(s^2+9)(0) - 3(2s)}{(s^2+9)^2} = \dfrac{-6s}{(s^2+9)^2}$

$F''_{(s)} = \dfrac{(s^2+9)^2(-6) - (-6s)\left(2(s^2+9)(2s)\right)}{(s^2+9)^4}$

$$\Rightarrow F''(s) = \frac{\cancel{(s^2+9)}\left[(s^2+9)(-6)+24s^2\right]}{(s^2+9)^{\cancel{4}}} = \frac{-6s^2-54+24s^2}{(s^2+9)^3}$$

$$F''(s) = \frac{18s^2-54}{(s^2+9)^3}$$

③ نضرب الناتج السابق

$$(-1)^2 = 1$$

$$\Rightarrow \mathcal{L}\, t^2 \sin 3t = \frac{18s^2-54}{(s^2+9)^3} \times 1$$

$$\boxed{\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}}$$

1) ناخذ $\mathcal{L} f(t) = F(s)$

② تقسم الدالة $F(s)$ على S

او نضرب بـ $\left(\frac{1}{s}\right)$

1) $\mathcal{L} \int_0^t e^{7u}\,du \Rightarrow \mathcal{L}\, e^{7t} = \frac{1}{s-7} = F(s)$

ثم نضرب الدالة $F(s)$ بـ $\frac{1}{s}$ $\Leftarrow$

$$F(s) = \frac{1}{s-7} \times \frac{1}{s} = \frac{1}{s(s-7)}$$

2) $\mathcal{L} \int_0^t \cos 6t\, dt$

$$\mathcal{L} \cos 6t = \frac{s}{s^2+36} = F(s) \Rightarrow \frac{F(s)}{s} = \frac{\cancel{s}}{s^2+36} \times \frac{1}{\cancel{s}}$$

$$\therefore \mathcal{L} \int_0^t \cos 6t\, dt = \frac{1}{s^2+36}$$

$$\boxed{5)\ \mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)}$$ (Time Scaling property)

① $\mathcal{L} f(t) = F(s)$ ② نعوض بدل كل S بـ $\frac{s}{a}$ ③ نضرب النتائج بـ $\frac{1}{a}$

Ex $\mathcal{L} \cos 3t = \frac{s}{s^2+9}$

or $f(at) = f(3t) = \cos 3t$

$f(t) = \cos t \implies$ ① $\mathcal{L} f(t) = \mathcal{L} \cos t = \frac{s}{s^2+1}$

② نبدل كل S بـ $\frac{s}{3}$ $\implies \frac{\frac{s}{3}}{\left(\frac{s}{3}\right)^2+1} =$

$$\frac{\frac{s}{3}}{\frac{s^2}{9}+1} = \frac{\frac{s}{3}}{\frac{s^2+9}{9}} = \frac{\frac{9s}{3}}{s^2+9} = \frac{3s}{s^2+9}$$

③ نضرب النتائج بـ $\frac{1}{3}$ $\implies \frac{1}{3} \cdot \frac{3s}{s^2+9} = \frac{s}{s^2+9}$

6) First derivative:

$$\mathcal{L} f'(t) = s\,\mathcal{L} f(t) - f(0) = \boxed{sF(s) - f(0)}$$

$$\mathcal{L} f''(t) = s^2 \mathcal{L} f(t) - s f(0) - f'(0) = \boxed{s^2 F(s) - s f(0) - f'(0)}$$

$$\mathcal{L} f'''(t) = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$$

$$\mathcal{L} f''''(t) = s^4 F(s) - s^3 f(0) - s^2 f'(0) - s f''(0) - f'''(0)$$

$$\mathcal{L} f^{(n)}(t) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

Examples

(1) Find $F(s)$ if $f(t) = t\,e^{-2t}$

Solution

using $\mathcal{L}(t^n f(t)) = (-1)^n F^{(n)}(s)$

(1) $F(s) \Leftarrow \mathcal{L} f(t)$ نوجد

(2) نشتق $F(s)$ [عدد المرات = أس t = n]

(3) نضرب الناتج بـ $(-1)^n$

(1) $\mathcal{L}\, e^{-2t} = \dfrac{1}{s+2} = F(s)$

(2) $F'(s) = \dfrac{(s+2)(0) - 1(1)}{(s+2)^2} = \dfrac{-1}{(s+2)^2}$

(3) نضرب بـ (-1)

$\Rightarrow \mathcal{L}\, t e^{-2t} = \dfrac{-1}{(s+2)^2} \times -1 = \dfrac{1}{(s+2)^2}$

(2) Find $\mathcal{L}\, t \sin(2t)$

(1) $\mathcal{L} \sin 2t = \dfrac{2}{s^2+4} = F(s)$

(2) $F'(s) = \dfrac{(s^2+4)(0) - 2(2s)}{(s^2+4)^2} = \dfrac{-4s}{(s^2+4)^2}$

(3) نضرب بـ (-1)

$\Rightarrow \mathcal{L}\, t \sin(2t) = \dfrac{4s}{(s^2+4)^2}$

3) If $\mathcal{L}[f(t)] = \dfrac{s}{s^3+2}$ find $\mathcal{L}[e^{-t} f(2t)]$

Sol:

$\mathcal{L}\, e^{\mp at} f(t) = F(s \pm a) \Rightarrow$ (1) $\mathcal{L} f(t) = F(s)$

(2) نبدل كل s بـ $(s \pm a)$

$\therefore \mathcal{L} f(2t) = ?$

$$\mathcal{L} f(at) = \frac{1}{a} F\left(\frac{s}{a}\right) \Rightarrow \mathcal{L} f(2t) = \frac{1}{2} F\left(\frac{s}{2}\right)$$

$$\because F(s) = \frac{s}{s^3+2} \Rightarrow \mathcal{L} f(2t) = \frac{1}{2} \frac{\frac{s}{2}}{\left(\frac{s}{2}\right)^3+2}$$

$$= \frac{1}{4} \frac{s}{\frac{s^3}{8}+2} = \frac{1}{4} \frac{s}{\frac{s^3+16}{8}} = \frac{1}{4} \frac{8s}{s^3+16} = \frac{2s}{s^3+16}$$

$$\Rightarrow \mathcal{L}[e^{-t} f(2t)] = \frac{2(s+1)}{(s+1)^3+16}$$

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4) $\mathcal{L}[t^4 - 4t + 5 + 3\sin 2t]$

Sol:

$$\mathcal{L} t^4 - 4\mathcal{L} t + \mathcal{L} 5 + 3\mathcal{L} \sin 2t$$

$$= \frac{4!}{s^5} - 4\frac{1}{s^2} + \frac{5}{s} + 3\frac{2}{s^2+4} =$$

$$\frac{24}{s^5} - \frac{4}{s^2} + \frac{5}{s} + \frac{6}{s^2+4}$$

(5) $\mathcal{L}\{4t\cos 7t\}$

Solu $4\mathcal{L}\{t\cos 7t\} =$

(1) $\mathcal{L}(\cos 7t) = \dfrac{s}{s^2+49}$ (2) $F'(s) = \dfrac{(s^2+49)(1) - s(2s)}{(s^2+49)^2}$

$F'(s) = \dfrac{s^2+49-2s^2}{(s^2+49)^2} = \dfrac{-s^2+49}{(s^2+49)^2} \quad \times(-1)$

$= \dfrac{s^2-49}{(s^2+49)^2} \Rightarrow 4\mathcal{L}(t\cos 7t) = \dfrac{4(s^2-49)}{(s^2+49)^2}$

(6) Find $\mathcal{L}\cos^2 t$

Solu

$\because \cos^2 t = \dfrac{1+\cos 2t}{2} = \dfrac{1}{2}(1+\cos 2t)$

$\therefore \mathcal{L}\,\dfrac{1}{2}(1+\cos 2t) = \dfrac{1}{2}\left[\mathcal{L}(1) + \mathcal{L}\cos(2t)\right]$

$= \dfrac{1}{2}\left[\dfrac{1}{s} + \dfrac{s}{s^2+4}\right] = \dfrac{1}{2}\,\dfrac{s^2+4+s^2}{s(s^2+4)} = \dfrac{1}{2}\,\dfrac{2s^2+4}{s(s^2+4)} = \dfrac{2}{2}\,\dfrac{s^2+2}{s(s^2+4)}$

$= \dfrac{s^2+2}{s(s^2+4)}$

(8) $\mathcal{L} \int_0^t \sin 2u \, du$?

Solu $\mathcal{L} \int_0^t f(u)\, du = \dfrac{F(s)}{s}$ (1) $\mathcal{L} f(t) = F(s)$

$\therefore$ (1) $\mathcal{L} \sin 2t = \dfrac{2}{s^2+4} = F(s)$ | (2) $\dfrac{1}{s}$ نضرب

(2) $\dfrac{F(s)}{s} = \dfrac{2}{s^2+4} \times \dfrac{1}{s} = \dfrac{2}{s(s^2+4)}$

(9) $\mathcal{L}\{t^2 e^{2t}\}$

Solu (1) $\mathcal{L}\, e^{2t} = \dfrac{1}{s-2}$

(2) $F'(s) = \dfrac{-1}{(s-2)^2} \Rightarrow F''(s) = \dfrac{(s-2)^2(0) - (-1)\left[2(s-2)\right]}{(s-2)^4}$

$F''(s) \; \dfrac{2\cancel{(s-2)}}{(s-2)^{\cancel{4}\,3}} = \dfrac{2}{(s-2)^3}$ (3) $1 = (-1)^2$ نضرب

$\therefore \mathcal{L}\, t^2 e^{2t} = \dfrac{2}{(s-2)^3}$

(10) Find ($\mathcal{L}$) transform of the following functions:

(a) $f(t) = \int_0^t \cos(4t)\, dt$

Solu $f(t) = \cos(4t) \Rightarrow \mathcal{L} f(t) = \dfrac{s}{s^2+16} = F(s)$

$\therefore \dfrac{F(s)}{s} \Rightarrow \dfrac{\cancel{s}}{s^2+16} \times \dfrac{1}{\cancel{s}} = \dfrac{1}{s^2+16}$

ⓑ $f(t) = \int_0^t e^{2t} \cos(3t)\, dt$

Solu

$$\mathcal{L} \int_0^t f(t)\, dt = \frac{F(s)}{s}$$

① $\mathcal{L} f(t) \Rightarrow F(s)$

$\because f(t) = e^{2t} \cos(3t) \Rightarrow \mathcal{L} f(t) = \mathcal{L}\, e^{2t} \cos(3t)$

$$\mathcal{L}\, e^{2t} \left(\frac{s}{s^2+9}\right) = \frac{s-2}{(s-2)^2+9} = F(s)$$

$$\therefore \frac{F(s)}{s} = \frac{(s-2)}{s\left[(s-2)^2+9\right]}$$

c) $f(t) = \int_0^t t\, e^{-3t}\, dt$

sol

$$f(t) = t\, e^{-3t} \Rightarrow \mathcal{L} f(t) = \mathcal{L}\, t\, e^{-3t} = \mathcal{L}\, t \quad \frac{1}{s+3}$$

$\Rightarrow \dfrac{-1}{(s+3)^2} \Rightarrow (-1)$ نضرب

$$\Rightarrow \mathcal{L}\, t\, e^{-3t} = \frac{1}{(s+3)^2} = F(s)$$

$$\therefore \frac{F(s)}{s} = \frac{1}{s(s+3)^2}$$

d) $f(t) = \int_0^t \sin(at)\cos(at)\,dt$

Solu

We have $\sin 2t = 2\sin t\cos t$

$\therefore \sin(2at) = 2\sin(at)\cos(at) \quad \div 2$

$\frac{\sin(2at)}{2} = \sin(at)\cos(at)$

$\Rightarrow \quad f(t) = \int_0^t \frac{\sin 2at}{2}\,dt$

$\therefore \mathcal{L}\ \frac{\sin(2at)}{2} = \frac{1}{2}\mathcal{L}\sin(2at) = \frac{1}{\cancel{2}}\ \frac{\cancel{2}a}{s^2+4a^2}$

$= \frac{a}{s^2+4a^2} = F(s)$

$\therefore \frac{F(s)}{s} = \frac{a}{s(s^2+4a^2)}$

H.w

$\mathcal{L}\{e^{4t}\cosh(5t)\}$, $\mathcal{L}\{e^{-2t}(3\cos 6t - 5\sin 6t)\}$

The Inverse Laplace Transform

The inverse Laplace transform of the function $F(s)$ is the unique function $f(t)$ that is continous on $[0,\infty)$ and satisfies $\mathcal{L}\{f(t)\} = F(s)$, we will use the notation

$\{\mathcal{L}^{-1}[F(s)]\}$ to denote the inverse laplace transform of $F(s)$, and we will write:

$$\mathcal{L}^{-1}[F(s)] = f(t)$$

Linearity of the Inverse Transform

let $f_1(t)$ and $f_2(t)$ be the inverse laplace transforms of $F_1(s)$ and $F_2(s)$ respectively and let C be aconstant we have:

$$\mathcal{L}^{-1}[C F_1(s)] = C\mathcal{L}^{-1}[F_1(s)] = C f_1(t)$$

$$\mathcal{L}^{-1}[F_1(s) \mp F_2(s)] = \mathcal{L}^{-1}[F_1(s)] \mp \mathcal{L}^{-1}[F_2(s)] = f_1(t) \mp f_2(t)$$

$$\mathcal{L}^{-1}[C_1 F_1(s) \mp C_2 F_2(s)] = C_1\mathcal{L}^{-1}[F_1(s)] \mp C_2\mathcal{L}^{-1}[F_2(s)] = C_1 f_1(t) \mp C_2 f_2(t)$$

ples

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$$

$$\boxed{\mathcal{L}\, t^n = \frac{n!}{s^{n+1}}}$$

② $\mathcal{L}^{-1}\left[\frac{3}{s}\right] = 3\mathcal{L}^{-1}\frac{1}{s} = 3$

③ $\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$

④ $\mathcal{L}^{-1}\left[\frac{7}{s^3}\right] = \frac{7}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} = \frac{7}{2}t^2$

⑤ $\mathcal{L}^{-1}\left[\frac{3s+4}{s^2+4}\right] = \mathcal{L}^{-1}\frac{3s}{s^2+4} + \mathcal{L}^{-1}\frac{4}{s^2+4} = 3\mathcal{L}^{-1}\frac{s}{s^2+4} + 2\mathcal{L}^{-1}\frac{2}{s^2+4}$

$= 3\cos 2t + 2\sin 2t$

⑥ $\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$

⑦ $\mathcal{L}^{-1}\left\{\frac{2}{(s+5)^4}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{(s+5)^4}\right\} = 2e^{-5t}\mathcal{L}^{-1}\frac{1}{s^4} = \frac{2}{3!}e^{-5t}\mathcal{L}^{-1}\frac{3!}{s^4}$

$= \frac{1}{3}e^{-5t}t^3$

⑧ $\mathcal{L}^{-1}\left\{\frac{6}{2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s-\frac{3}{2}}\right\} = 3e^{\frac{3}{2}t}$

⑨ $\mathcal{L}^{-1}\left\{\frac{3+4s}{9s^2-16}\right\} = \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{3}{s^2-\frac{16}{9}}\right\} + \frac{4}{9}\mathcal{L}^{-1}\left\{\frac{s}{s^2-\frac{16}{9}}\right\}$

$= \frac{1}{4}\sinh\frac{4}{3}t + \frac{4}{9}\cosh\frac{4}{3}t$

$$ (8)\ \mathcal{L}^{-1}\left\{\frac{6}{2S-3}\right\} = \mathcal{L}^{-1}\left\{\frac{6}{2(S-\frac{3}{2})}\right\} = \mathcal{L}^{-1}\frac{3}{S-\frac{3}{2}} = 3\mathcal{L}^{-1}\frac{1}{S-\frac{3}{2}} = 3e^{+\frac{3}{2}t} $$

$$ 9)\ \mathcal{L}^{-1}\left\{\frac{3+4S}{9S^2-16}\right\} = \mathcal{L}^{-1}\left\{\frac{3+4S}{9(S^2-\frac{16}{9})}\right\} = \frac{1}{9}\mathcal{L}^{-1}\frac{3+4S}{S^2-\frac{16}{9}} $$

$$ = \frac{3}{9}\mathcal{L}^{-1}\frac{1}{S^2-\frac{16}{9}} + \frac{4}{9}\mathcal{L}^{-1}\frac{S}{S^2-\frac{16}{9}} $$

$$ = \frac{3}{9}\mathcal{L}^{-1}\frac{\frac{4}{3}\big/\frac{4}{3}}{S^2-\frac{16}{9}} + \frac{4}{9}\mathcal{L}^{-1}\frac{S}{S^2-\frac{16}{9}} = \frac{3}{9}\cdot\frac{3}{4}\mathcal{L}^{-1}\frac{\frac{4}{3}}{S^2-\frac{16}{9}} + \frac{4}{9}\mathcal{L}^{-1}\frac{S}{S^2-\frac{16}{9}} $$

$$ \frac{1}{4}\sinh\frac{4}{3}t + \frac{4}{9}\cosh\frac{4}{3}t $$

$$ 10)\ \mathcal{L}^{-1}\left\{\frac{3S-5}{4(S^2-S+\frac{37}{4})}\right\} = \frac{1}{4}\mathcal{L}^{-1}\frac{3S-5}{S^2-S+\frac{37}{4}} $$

اضافة وطرح
نصف مربع معامل "S"
معامل $S = -1$
$(-\frac{1}{2})^2 = +\frac{1}{4}$

$$ = \frac{1}{4}\mathcal{L}^{-1}\frac{3S-5}{(S^2-S+\frac{1}{4})+\frac{37}{4}-\frac{1}{4}} = \frac{1}{4}\mathcal{L}^{-1}\frac{3S-5}{(S-\frac{1}{2})^2+9} $$

$$ \frac{1}{4}\mathcal{L}^{-1}\frac{3S}{(S-\frac{1}{2})^2+9} + \frac{1}{4}\mathcal{L}^{-1}\frac{-5}{(S-\frac{1}{2})^2+9} = \frac{3}{4}\mathcal{L}^{-1}\frac{S}{(S-\frac{1}{2})^2+9} - \frac{5}{4}\mathcal{L}^{-1}\frac{1}{(S-\frac{1}{2})^2+9} $$

$$ \frac{3}{4}\mathcal{L}^{-1}\frac{(S-\frac{1}{2})+\frac{1}{2}}{(S-\frac{1}{2})^2+9} - \frac{5}{4}\mathcal{L}^{-1}\frac{1}{(S-\frac{1}{2})^2+9}\cdot\frac{3}{3} $$

$$ \frac{3}{4}\mathcal{L}^{-1}\frac{S-\frac{1}{2}}{(S-\frac{1}{2})^2+9} + \frac{3}{4}\mathcal{L}^{-1}\frac{\frac{1}{2}}{(S-\frac{1}{2})^2+9} - \frac{5}{12}\mathcal{L}^{-1}\frac{3}{(S-\frac{1}{2})^2+9} $$

$$\frac{3}{4} e^{\frac{1}{2}t} \mathcal{L}^{-1} \frac{s}{s^2+9} + \frac{1}{8} e^{\frac{1}{2}t} \mathcal{L}^{-1} \frac{3}{s^2+9} - \frac{5}{12} e^{\frac{1}{2}t} \mathcal{L}^{-1} \frac{3}{s^2+9}$$

$$\frac{3}{4} e^{\frac{1}{2}t} \cos(3t) + \frac{1}{8} e^{\frac{1}{2}t} \sin(3t) - \frac{5}{12} e^{\frac{1}{2}t} \sin 3t$$

$$\frac{3}{4} e^{\frac{1}{2}t} \cos(3t) - \frac{7}{24} e^{\frac{1}{2}t} \sin 3t$$

(15) $\mathcal{L}^{-1}\left\{ \frac{6}{2s-3} - \frac{3+4s}{9s^2-16} + \frac{8-6s}{16s^2+9} \right\}$

$$\mathcal{L}^{-1}\left\{ \frac{6}{2(s-\frac{3}{2})} - \frac{3+4s}{9(s^2-\frac{16}{9})} + \frac{8-6s}{16(s^2+\frac{9}{16})} \right\}$$

$$= \mathcal{L}^{-1}\left[\frac{3}{s-\frac{3}{2}} - \frac{3}{9(s^2-\frac{16}{9})} - \frac{4s}{9(s^2-\frac{16}{9})} + \frac{8}{16(s^2+\frac{9}{16})} - \frac{6s}{16(s^2+\frac{9}{16})} \right]$$

$$= 3\mathcal{L}^{-1} \frac{1}{s-\frac{3}{2}} - \frac{1}{3}\mathcal{L}^{-1} \frac{1}{s^2-\frac{16}{9}} - \frac{4}{9}\mathcal{L}^{-1} \frac{s}{s^2-\frac{16}{9}} + \frac{1}{2}\mathcal{L}^{-1} \frac{1}{s^2+\frac{9}{16}} - \frac{3}{8}\mathcal{L}^{-1} \frac{s}{s^2+\frac{9}{16}}$$

$$= 3e^{\frac{3}{2}t} - \frac{1}{3}\mathcal{L}^{-1} \frac{\frac{4}{3}\cdot\frac{3}{4}}{s^2-\frac{16}{9}} - \frac{4}{9}\cosh(\tfrac{4}{3}t) + \frac{1}{2}\mathcal{L}^{-1} \frac{\frac{3}{4}\cdot\frac{4}{3}}{s^2+\frac{9}{16}} - \frac{3}{8}\cos(\tfrac{3}{4}t)$$

$$= 3e^{\frac{3}{2}t} - \frac{1}{4}\sinh(\tfrac{4}{3}t) - \frac{4}{9}\cosh(\tfrac{4}{3}t) + \frac{2}{3}\sinh(\tfrac{3}{4}t) - \frac{3}{8}\cos(\tfrac{3}{4}t)$$

$$\left\{\frac{3s-5}{4(s^2-s+\frac{37}{4})}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{3s-5}{(s^2-s+\frac{1}{4})+9}\right\} = \frac{1}{4}\mathcal{L}^{-1}\frac{3s-5}{(s-\frac{1}{2})^2+3^2}$$

$$= \frac{1}{4}\mathcal{L}^{-1}\frac{3s}{(s-\frac{1}{2})^2+3^2} - \frac{5}{4}\mathcal{L}^{-1}\frac{1}{(s-\frac{1}{2})^2+3^2} =$$

$$= \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{(s-\frac{1}{2})+\frac{1}{2}}{(s-\frac{1}{2})^2+9}\right\} - \frac{5}{12}\mathcal{L}^{-1}\left\{\frac{3}{(s-\frac{1}{2})^2+3^2}\right\}$$

$$= \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{(s-\frac{1}{2})}{(s-\frac{1}{2})^2+9} + \frac{3}{8}\,\frac{1}{(s-\frac{1}{2})^2+9}\right\} - \frac{5}{12}e^{\frac{1}{2}t}\sin 3t$$

$$= \frac{3}{4}e^{\frac{1}{2}t}\cos 3t + \frac{1}{8}e^{\frac{1}{2}t}\sin 3t - \frac{5}{12}e^{\frac{1}{2}t}\sin 3t$$

$$= \frac{3}{4}e^{\frac{1}{2}t}\cos 3t - \frac{7}{24}e^{\frac{1}{2}t}\sin 3t = \frac{1}{4}e^{\frac{1}{2}t}\left(3\cos 3t - \frac{7}{6}\sin 3t\right) = f(t)$$

---

(11) $$\mathcal{L}^{-1}\left\{\frac{3s-5}{4s^2+4s+1}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{3s-5}{(s^2+s+\frac{1}{4})}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{3s-5}{(s+\frac{1}{2})^2}\right\}$$

$$= \frac{3}{4}\mathcal{L}^{-1}\frac{s}{(s+\frac{1}{2})^2} - \frac{5}{4}\mathcal{L}^{-1}\frac{1}{(s+\frac{1}{2})^2} = \frac{3}{4}\mathcal{L}^{-1}\frac{(s+\frac{1}{2})-\frac{1}{2}}{(s+\frac{1}{2})^2} - \frac{5}{4}\mathcal{L}^{-1}\frac{1}{(s+\frac{1}{2})^2}$$

$$= \frac{3}{4}\mathcal{L}^{-1}\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2} - \frac{3}{8}\mathcal{L}^{-1}\frac{1}{(s+\frac{1}{2})^2} - \frac{5}{4}\mathcal{L}^{-1}\frac{1}{(s+\frac{1}{2})^2}$$

$$= \frac{3}{4}\mathcal{L}^{-1}\frac{1}{s+\frac{1}{2}} - \frac{13}{8}\mathcal{L}^{-1}\frac{1}{(s+\frac{1}{2})^2} = \frac{3}{4}e^{-\frac{1}{2}t} - \frac{13}{8}e^{-\frac{1}{2}t}\,t$$

$$= \frac{1}{4}e^{-\frac{1}{2}t}\left(3 - \frac{13}{2}t\right)$$

$$\frac{3}{s^2+4s+13} = \mathcal{L}^{-1}\left\{\frac{3}{(s^2+4s+4)-4+13}\right\} = \mathcal{L}^{-1}\frac{3}{(s+2)^2+9}$$

$$= e^{-2t}\mathcal{L}^{-1}\frac{3}{s^2+9} = e^{-2t}\sin 3t$$

(13) $\mathcal{L}^{-1}\frac{1}{s(s^2+4)}$

Solu: using $\int_0^t f(u)du = \frac{F(s)}{s}$

$$\therefore \mathcal{L}^{-1}\frac{\frac{1}{s^2+4}}{s} = \mathcal{L}^{-1}\frac{F(s)}{s} \Rightarrow F(s) = \frac{1}{s^2+4}$$

$$= \mathcal{L}^{-1}F(s) = f(u) \Rightarrow \mathcal{L}^{-1}\frac{1}{s^2+4} = \frac{1}{2}\mathcal{L}^{-1}\frac{2}{s^2+4} = \frac{1}{2}\sin 2u = f(u)$$

$$= \mathcal{L}^{-1}F(s) = f(u) \Rightarrow \int_0^t f(u)du = \int_0^t \frac{1}{2}\sin 2u\,du = -\frac{1}{4}\int_0^t -2\sin 2u\,du$$

$$-\frac{1}{4}\cos 2u\Big|_0^t = \frac{1}{4}(1-\cos 2t)$$

$$\therefore \mathcal{L}^{-1}\frac{1}{s(s^2+4)} = \frac{1}{4}(1-\cos 2t)$$

---

(14) Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$ since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

we have $\mathcal{L}^{-1}\frac{F(s)}{s} = \int_0^t f(u)du$

$$\therefore \mathcal{L}^{-1}\frac{1}{s(s^2+1)} = \int_0^t \sin u\,du = 1-\cos t$$

$\Longrightarrow$

$$\frac{1}{s^2(s^2+1)} = \int_0^t (1-\cos u)\,du = t - \sin t$$

and $$\mathcal{L}^{-1} \frac{1}{s^3(s^2+1)} = \int_0^t (u - \sin u)\,du = \frac{t^2}{2} + \cos t - 1$$

$$f(t) = \frac{t^2}{2} + \cos t - 1$$

note: to check your solution take

$\mathcal{L}\{f(t)\} = F(s)$ means!

$$\mathcal{L}\left\{\frac{t^2}{2} + \cos t - 1\right\} = \frac{1}{2}\,\frac{2!}{s^3} + \frac{s}{s^2+1} - \frac{1}{s} = \frac{1}{s^3} + \frac{s}{s^2+1} - \frac{1}{s}$$

$$F(s) = \frac{s^2+1+s^4-s^2(s^2+1)}{s^3(s^2+1)} = \frac{\cancel{s^2}+1+\cancel{s^4}-\cancel{s^4}-\cancel{s^2}}{s^3(s^2+1)} = \frac{1}{s^3(s^2+1)} = F(s)$$

(15) $$\mathcal{L}^{-1}\left\{\frac{6}{2s-3} - \frac{3+4s}{9s^2-16} + \frac{8-6s}{16s^2+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3}{s-\frac{3}{2}} - \frac{3}{9(s^2-\frac{16}{9})} - \frac{4}{9}\,\frac{s}{(s^2-\frac{16}{9})} + \frac{8}{16(s^2+\frac{9}{16})} - \frac{6}{16}\,\frac{s}{(s^2+\frac{9}{16})}\right\}$$

$$= \mathcal{L}^{-1}\frac{3}{s-\frac{3}{2}} - \frac{1}{3}\mathcal{L}^{-1}\frac{1}{s^2-\frac{16}{9}} - \frac{4}{9}\mathcal{L}^{-1}\frac{s}{s^2-\frac{16}{9}} + \frac{1}{2}\mathcal{L}^{-1}\frac{1}{s^2+\frac{9}{16}} - \frac{3}{8}\mathcal{L}^{-1}\frac{s}{s^2+\frac{9}{16}}$$

$$3e^{\frac{3}{2}t} - \frac{1}{4}\sinh\frac{4}{3}t - \frac{4}{9}\cosh\frac{4}{3}t + \frac{2}{3}\sin\frac{3}{4}t - \frac{3}{8}\cos\frac{3}{4}t$$

1) $\mathcal{L}^{-1}\dfrac{1}{2S-5} = \mathcal{L}^{-1}\dfrac{1}{2(S-\frac{5}{2})} = 2\mathcal{L}^{-1}\dfrac{1}{S-\frac{5}{2}} = 2e^{+\frac{5}{2}t}$

2) $\mathcal{L}^{-1}\dfrac{12}{4-3S} = \mathcal{L}^{-1}\dfrac{12}{-3S+4} = \mathcal{L}^{-1}\dfrac{12}{-3(S-\frac{4}{3})} = \mathcal{L}\dfrac{-4}{S-\frac{4}{3}}$

$-4\mathcal{L}^{-1}\dfrac{1}{S-\frac{4}{3}} = -4e^{+\frac{4}{3}t}$

3) $\mathcal{L}^{-1}\dfrac{3(S^2-1)^2}{2S^5} = \dfrac{3}{2}\mathcal{L}^{-1}\dfrac{(S^2-1)^2}{S^5} = \dfrac{3}{2}\mathcal{L}^{-1}\dfrac{S^4-2S^2+1}{S^5}$

$\dfrac{3}{2}\mathcal{L}^{-1}\dfrac{S^4}{S^5} - \dfrac{2S^2}{S^5} + \dfrac{1}{S^5} = \dfrac{3}{2}\mathcal{L}^{-1}\dfrac{1}{S} - \dfrac{2}{S^3} + \dfrac{1}{S^5}$

$\dfrac{3}{2}(1 - t^2 + \dfrac{1}{24}t^4)$

4) $\mathcal{L}^{-1}\dfrac{4S-18}{9-S^2} = \mathcal{L}^{-1}\dfrac{4S-18}{-S^2+9} = -\mathcal{L}^{-1}\left(\dfrac{4S-18}{S^2-9}\right)$

$-\mathcal{L}^{-1}\left(\dfrac{4S}{S^2-9} - \dfrac{18}{S^2-9}\right) = -4\mathcal{L}^{-1}\dfrac{S}{S^2-9} + \mathcal{L}^{-1}\dfrac{6\times3}{S^2-9}$

$-4\cosh 3t + 6\sinh 3t$

Partial fraction method

طريقة الكسور الجزئية

$$\frac{f(s)}{g(s)} \qquad g(s) \neq 0$$

1) $g(s) = (as \mp b)(cs \mp D)(es \mp f) \dots$

$$\frac{A}{(as \mp b)} + \frac{B}{(cs \mp d)} + \frac{C}{(es \mp f)} + \dots$$

Ex

$$\frac{\quad}{(s-2)(s+4)(s-1)} = \frac{A}{s-2} + \frac{B}{s+4} + \frac{C}{s-1}$$

2) $g(s) = s^n (as \mp b)$

$$\frac{A}{s^n} + \frac{B}{s^{n-1}} + \frac{C}{s^{n-2}} + \dots \frac{\quad}{s^1} + \frac{k}{as \mp b}$$

Ex

$$\frac{\quad}{s^3(s+5)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s+5}$$

3) $g(s) = (as^2 + bs + D)(es \mp k)$

$$\frac{As+B}{as^2+bs+D} + \frac{C}{es \mp k}$$

Ex

$$\frac{\quad}{(3s^2-s+4)(s+3)} = \frac{As+B}{3s^2-s+4} + \frac{C}{s+3}$$

Partial Fraction method

Ex 16 Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2-5s+6}\right\}$

Solution:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-5s+6}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)(s-2)}\right\} =$$

$$\mathcal{L}^{-1}\left(\frac{A}{s-3} + \frac{B}{s-2}\right) = \mathcal{L}^{-1} \frac{A(s-2) + B(s-3)}{(s-3)(s-2)}$$

$$= \mathcal{L}^{-1} \frac{As - 2A + Bs - 3B}{(s-3)(s-2)} = \mathcal{L}^{-1} \frac{s(A+B) + (-2A - 3B)}{(s-3)(s-2)}$$

$$\frac{1}{(s-3)(s-2)} = \frac{s(A+B) + (-2A-3B)}{(s-3)(s-2)}$$

$$-2A - 3B = 1 \quad \text{---} (1)$$

$$A + B = 0 \Rightarrow \boxed{A = -B} \quad \text{تعويض معادلة } (1)$$

$$-2(-B) - 3B = 1 \Rightarrow 2B - 3B = 1 \Rightarrow \boxed{B = -1}, \; A = 1$$

$$\Rightarrow \mathcal{L}^{-1} \frac{1}{s-3} + \frac{-1}{s-2} = e^{3t} - e^{2t}$$

Ex. Find $\mathcal{L}^{-1}\left\{\dfrac{3S+7}{S^2-2S-3}\right\}$

Solution $\mathcal{L}^{-1}\left\{\dfrac{3S+7}{S^2-2S-3}\right\} = \mathcal{L}^{-1}\left\{\dfrac{3S+7}{(S-3)(S+1)}\right\} = \mathcal{L}^{-1}\left(\dfrac{A}{S-3} + \dfrac{B}{S+1}\right)$

$= \mathcal{L}^{-1}\dfrac{A(S+1)+B(S-3)}{(S-3)(S+1)} = \mathcal{L}^{-1}\dfrac{AS+A+BS-3B}{(S-3)(S+1)}$

$= \mathcal{L}^{-1}\dfrac{S(A+B)+(A-3B)}{(S-3)(S+1)} \Rightarrow 3S+7 = S(A+B)+(A-3B)$

$A+B=3$
$A-3B=7$ بالطرح $\Rightarrow$ $A+B=3$, $-A+3B=-7$ $>$ $4B=-4 \Rightarrow \boxed{B=-1}$

$\boxed{A=4}$

$\therefore \mathcal{L}^{-1}\dfrac{4}{S-3} + \dfrac{-1}{S+1} = 4e^{3t} - e^{-t}$

or

$$\mathcal{L}^{-1}\frac{3s+7}{s^2-2s-3}=\mathcal{L}^{-1}\frac{3s+7}{(s-3)(s+1)}=\mathcal{L}^{-1}\frac{A}{s-3}+\frac{B}{s+1}$$

$$A=\frac{3s+7}{\cancel{(s-3)}(s+1)}\cancel{(s-3)}\Big|_{s=3}=\frac{16}{4}=\boxed{4=A}$$

$$B=\frac{3s+7}{(s-3)\cancel{(s+1)}}\cancel{(s+1)}\Big|_{s=-1}=\frac{4}{-4}=\boxed{-1=B}$$

$$\therefore\mathcal{L}^{-1}\frac{4}{s-3}+\frac{-1}{s+1}=4e^{3t}-e^{-t}$$

Ex Solve $\mathcal{L}^{-1} \dfrac{2s^2-16}{s^3 - 16s}$ Using Partial Fraction

Sol:

$$\mathcal{L}^{-1} \frac{2s^2-16}{s(s^2-16)} = \mathcal{L}^{-1} \frac{2s^2-16}{s(s-4)(s+4)} = \mathcal{L}^{-1} \frac{A}{s} + \frac{B}{s-4} + \frac{C}{s+4}$$

$$= \mathcal{L}^{-1} \frac{A(s^2-16) + Bs(s+4) + Cs(s-4)}{s(s-4)(s+4)}$$

$$= \mathcal{L}^{-1} \frac{As^2 - 16A + Bs^2 + 4Bs + Cs^2 - 4Cs}{s(s-4)(s+4)}$$

$A + B + C = 2$ --- (1)

$4B - 4C = 0$ --- (2)

$-16A = -16$ --- (3) $\Rightarrow \boxed{A = 1}$ Put in (1)

$B + C = 1 \Longrightarrow$

$4B - 4C = 0 \div 4 \Rightarrow \begin{matrix} B + C = 1 \\ B - C = 0 \end{matrix} \overset{+}{\Rightarrow} 2B = 1 \Rightarrow \boxed{B = \frac{1}{2}}, \boxed{C = \frac{1}{2}}$

$$\therefore \mathcal{L}^{-1} \frac{1}{s} + \frac{\frac{1}{2}}{s-4} + \frac{\frac{1}{2}}{s+4} = 1 + \frac{1}{2} e^{4t} + \frac{1}{2} e^{-4t}$$

$\underline{\text{or}}$ $\mathcal{L}^{-1} \dfrac{2s^2-16}{s(s-4)(s+4)} = \mathcal{L}^{-1} \dfrac{A}{s} + \dfrac{B}{s-4} + \dfrac{C}{s+4}$

$A = \dfrac{2s^2-16}{\cancel{s}(s-4)(s+4)} \cancel{s}\Big|_{s=0} = \dfrac{-16}{-16} = \boxed{1 = A}$

$B = \dfrac{2s^2-16}{s\cancel{(s-4)}(s+4)} \cancel{(s-4)}\Big|_{s=4} = \dfrac{16}{32} = \boxed{\dfrac{1}{2} = B}$

$C = \dfrac{2s^2-16}{s(s-4)\cancel{(s+4)}} \cancel{(s+4)}\Big|_{s=-4} = \dfrac{16}{32} = \boxed{\dfrac{1}{2} = C}$

$= \mathcal{L}^{-1} \dfrac{1}{s} + \dfrac{\frac{1}{2}}{s-4} + \dfrac{\frac{1}{2}}{s+4} = 1 + \dfrac{1}{2}e^{4t} + \dfrac{1}{2}e^{-4t}$

Q Find $\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+4)}\right)$ using Partial fraction method

Solution

$$\mathcal{L}^{-1}\frac{1}{s^2(s^2+4)} = \mathcal{L}^{-1}\left(\frac{A}{s^2}+\frac{B}{s}+\frac{Cs+D}{(s^2+4)}\right)$$

$$= \mathcal{L}^{-1}\frac{A(s^2+4)+Bs(s^2+4)+s^2(Cs+D)}{s^2(s^2+4)} =$$

$$= \mathcal{L}^{-1}\frac{As^2+4A+Bs^3+4Bs+Cs^3+Ds^2}{s^2(s^2+4)}$$

$$= \mathcal{L}^{-1}\frac{s^3(B+C)+s^2(A+D)+4Bs+4A}{s^2(s^2+4)}$$

$$\therefore \quad 1 = s^3(B+C)+s^2(A+D)+4Bs+4A$$

$B+C=0$

$A+D=0$

$4B=0 \Rightarrow \boxed{B=0} \Rightarrow \boxed{C=0}$

$4A=1 \Rightarrow \boxed{A=\frac{1}{4}} \Rightarrow \boxed{D=-\frac{1}{4}}$

$$= \mathcal{L}^{-1}\left(\frac{\frac{1}{4}}{s^2}+\frac{0}{s}+\frac{0s-\frac{1}{4}}{s^2+4}\right)$$

$$= \frac{1}{4}\mathcal{L}^{-1}\frac{1}{s^2} - \frac{1}{4}\mathcal{L}^{-1}\frac{1}{s^2+4}\cdot\frac{2}{2} = \frac{1}{4}\mathcal{L}^{-1}\frac{1}{s^2} - \frac{1}{8}\mathcal{L}^{-1}\frac{2}{s^2+4}$$

$$= \frac{1}{4}t - \frac{1}{8}\sin 2t = f(t)$$ (7)

Q $\mathcal{L}^{-1} \frac{1}{s^3(s^2+1)} = \mathcal{L}^{-1} \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{Ds+E}{s^2+1}$

$= \mathcal{L}^{-1} \frac{A(s^2+1) + Bs(s^2+1) + Cs^2(s^2+1) + s^3(Ds+E)}{s^3(s^2+1)}$

$= \mathcal{L}^{-1} \frac{As^2 + A + Bs^3 + Bs + Cs^4 + Cs^2 + Ds^4 + Es^3}{s^3(s^2+1)}$

$= \mathcal{L}^{-1} \frac{s^4(C+D) + s^3(B+E) + s^2(A+C) + Bs + A}{s^3(s^2+1}$

$= C+D=0$

$B+E=0$

$A+C=0$

$\boxed{B=0} \Rightarrow \boxed{E=0}$

$\boxed{A=1} \Rightarrow \boxed{C=-1} \Rightarrow \boxed{D=1}$

$= \mathcal{L}^{-1} \frac{1}{s^3} + \frac{0}{s^2} + \frac{-1}{s} + \frac{s+0}{s^2+1}$

$= \mathcal{L}^{-1} \frac{1}{s^3} - \frac{1}{s} + \frac{s}{s^2+1} = \frac{1}{2}t^2 - 1 + \cos t = f(t)$

$$\mathcal{L}^{-1}\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \mathcal{L}^{-1}\frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

$$= \mathcal{L}^{-1}\frac{(As+B)(s^2+2s+5)+(Cs+D)(s^2+2s+2)}{(s^2+2s+2)(s^2+2s+5)}$$

$$= \mathcal{L}^{-1}\frac{As^3+2As^2+5As+Bs^2+2Bs+5B+Cs^3+2Cs^2+2Cs+Ds^2+2Ds+2D}{(s^2+2s+2)(s^2+2s+5)}$$

$$= \mathcal{L}^{-1}\frac{s^3(A+C)+s^2(2A+B+2C+D)+s(5A+2B+2C+2D)+(5B+2D)}{(s^2+2s+2)(s^2+2s+5)}$$

$A+C=0$ --- (1) $\Rightarrow \boxed{A=-C}$ put in (2) and (3)

$2A+B+2C+D=1$ --- (2)

$5A+2B+2C+2D=2$ -- (3)

$5B+2D=3$ -- (4) $\Rightarrow \boxed{B=\frac{3-2D}{5}}$ put in (2)

from (2) $-\cancel{2C}+\frac{3-2D}{5}+\cancel{2C}+D=1 \Rightarrow \boxed{D=\frac{2}{3}}$

$\boxed{B=\frac{1}{3}}$

(2) $\longrightarrow$ $2A+\frac{1}{3}+2C+\frac{2}{3}=1$

(3) $\longrightarrow$ $5A+\frac{2}{3}+2C+\frac{4}{3}=2$

$\therefore 2A+2C=0$

$5A+2C=0$ $\Rightarrow \boxed{A=C=0}$

$$= \mathcal{L}^{-1}\left(\frac{\frac{1}{3}}{s^2+2s+2} + \frac{\frac{2}{3}}{s^2+2s+5}\right)$$

$$= \mathcal{L}^{-1}\frac{\frac{1}{3}}{(s^2+2s+1)-1+2} + \mathcal{L}^{-1}\frac{\frac{2}{3}}{(s^2+2s+1)-1+5}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\frac{1}{(s+1)^2+1} + \frac{2}{3}\mathcal{L}^{-1}\frac{1}{(s+1)^2+4}$$

$$= \frac{1}{3}e^{-t}\mathcal{L}^{-1}\frac{1}{s^2+1} + \frac{2}{3}e^{-t}\mathcal{L}^{-1}\frac{1}{s^2+4}$$

$$= \frac{1}{3}e^{-t}\mathcal{L}^{-1}\frac{1}{s^2+1} + \frac{1}{3}e^{-t}\mathcal{L}^{-1}\frac{2}{s^2+4}$$

$$= \frac{1}{3}e^{-t}\sin t + \frac{1}{3}e^{-t}\sin 2t$$

"Problems"

1- $\dfrac{s^2+6s+6}{(s+1)(s^2+4s+3)}$

2- $\dfrac{12s-28}{(s-5)(s+3)(s-1)}$

3- $\dfrac{3s^2+8s+9}{(s+3)(s+1)^2}$

4- $\dfrac{s^3+4s^2+4s-5}{(s+3)(s+1)^3}$

5- $\dfrac{2s^2}{(s^2+1)(s-1)^2}$

6- $\dfrac{s}{s^2+6s+1}$

7- $\dfrac{s+1}{s^2+s+1}$

8- $\left\{ \dfrac{12}{4-3s} \right\}$

9- $\dfrac{1}{2s-5}$

10- $\left\{ \dfrac{3(s^2-1)^2}{2s^5} + \dfrac{4s-18}{9-s^2} \right.$

11- $\dfrac{s}{(s+1)^5}$

12- $\dfrac{3s-14}{s^2-4s+8}$

13- $\dfrac{8s+20}{s^2-12s+32}$

1- $\mathcal{L}^{-1} \dfrac{S^2+6S+6}{(S+1)(S^2+4S+3)} = \mathcal{L}^{-1} \dfrac{S^2+6S+6}{(S+1)(S+3)(S+1)} = \mathcal{L}^{-1} \dfrac{S^2+6S+6}{(S+1)^2(S+3)}$

$\mathcal{L}^{-1} \dfrac{A}{(S+1)^2} + \dfrac{B}{(S+1)} + \dfrac{C}{(S+3)}$ - - - - -

(4) $\mathcal{L}^{-1} \dfrac{S^3+4S^2+4S-5}{(S+3)(S+1)^3} = \mathcal{L}^{-1} \dfrac{A}{(S+1)^3} + \dfrac{B}{(S+1)^2} + \dfrac{C}{(S+1)} + \dfrac{D}{S+3}$

5) $\mathcal{L}^{-1} \dfrac{2S^2}{(S^2+1)(S-1)^2} = \mathcal{L}^{-1} \dfrac{AS+B}{S^2+1} + \dfrac{C}{(S-1)^2} + \dfrac{D}{(S-1)}$

6) $\mathcal{L}^{-1} \dfrac{S}{S^2+6S+1}$ (اكمال مربع) اضافه وطرح مربع نصف معامل S

$\mathcal{L}^{-1} \dfrac{S}{(S^2+6S+9)+1-9} = \mathcal{L} \dfrac{S}{(S+3)^2-8} \Rightarrow \mathcal{L} \dfrac{S+3-3}{(S+3)^2-8}$

$\mathcal{L} \dfrac{S+3}{(S+3)^2-8} - \dfrac{3}{(S+3)^2-8}$

12) $\mathcal{L}^{-1} \dfrac{S}{(S+1)^5} \Rightarrow \mathcal{L}^{-1} \dfrac{S+1-1}{(S+1)^5} \Rightarrow \mathcal{L}^{-1} \dfrac{S+1}{(S+1)^5} - \dfrac{1}{(S+1)^5}$

$\mathcal{L}^{-1} \dfrac{1}{(S+1)^4} - \dfrac{1}{(S+1)^5}$

Solution Of The Differential Equations Using Laplace Transform

The laplace transform is amethod of solving ordinary differential equations (ODE) and intial value problem.

Befor we solve any example we must first consider the laplace transform of derivatives.

Laplace Transform Of Derivatives

The transforms of the first and second derivatives of $f(t)$ satisfy

$$\mathcal{L}\, f'(t) = s\mathcal{L}\{f(t)\} - f(0) = sF(s) - f(0)$$

$$\mathcal{L}\, f''(t) = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) = s^2F(s) - sf(0) - f'(0)$$

and the higher order derivative [the laplace trans. of the derivative $f^{(n)}(t)$] satisfies;

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

...iffferential Equations, Initial Value problems

...we shall now discuss how the laplace transform method solves ODEs and intial value problems, the procedure is best illustrated with anexample :-

Example: Solve the (I.v.p.) by laplace Trans,

$y'' - 3y' - 10y = 2$ with $y(0)=1$, $y'(0)=2$

* Structure of Solution *

* Take laplace transform on both sides, You will gen an algebraic equation for Y
Solve the equation to get $Y_{(s)}$
Take inverse transform to get $y(t) = \mathcal{L}^{-1}\{Y_{(s)}\}$

Solution:

1- take laplace transform on both sides
let $\mathcal{L}\{y(t)\} = Y_{(s)}$

$\mathcal{L}\{y''(t)\} - 3\mathcal{L}\{y'(t)\} - 10\mathcal{L}\{y(t)\} = \mathcal{L}\{2\}$

$s^2 Y_{(s)} - s y(0) - y'(0) - 3[s Y_{(s)} - 10 y(0)] - 10 Y_{(s)} = \frac{2}{s}$

$s^2 Y_{(s)} - s - 2 - 3s Y_{(s)} + 3 - 10 Y_{(s)} = \frac{2}{s}$

Now we get an algebraic equation for $Y_{(s)}$

2- Solve it for $Y(s)$,

$$Y(s)(s^2-3s-10) = \frac{2}{s} + s - 1 \Rightarrow Y(s) = \frac{2+s^2-s}{s(s^2-3s-10)}$$

$$\therefore Y(s) = \frac{s^2-s+2}{s(s-5)(s+2)}$$

3- Take inverse laplace to get $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

The main technique here is partial fraction

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\frac{s^2-s+2}{s(s-5)(s+2)} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s-5} + \frac{C}{s+2}\right\}$$

$$= \frac{A(s^2-3s-10) + Bs(s+2) + Cs(s-5)}{s(s-5)(s+2)}$$

$$s^2-s+2 = As^2-3As-10A+Bs^2+2Bs+Cs^2-5Cs$$

$$A+B+C=1$$
$$-3A+2B-5C=-1$$
$$-10A=2 \Rightarrow \boxed{A=-\frac{1}{5}} \quad \boxed{B=\frac{22}{35}} \quad \boxed{C=\frac{4}{7}}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{-\frac{1}{5}}{s} + \frac{\frac{22}{35}}{s-5} + \frac{\frac{4}{7}}{s+2}\right)$$

$$\therefore y(t) = -\frac{1}{5} + \frac{22}{35}e^{5t} + \frac{4}{7}e^{-2t}$$

Ex Solve the intial value problem by laplace trans. $y'' - y' - 2y = e^{2t}$ $\quad y(0)=0,\ y'(0)=1$

Solu: take laplace trans. on both sides of th equation we get

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - \mathcal{L}\{2y\} = \mathcal{L}\{e^{2t}\}$$

$$s^2 Y(s) - 1 - s\,Y(s) - 2\,Y(s) = \frac{1}{s-2}$$

Solve it for Y

$$(s^2 - s - 2)\,Y(s) = \frac{1}{s-2} + 1 \Rightarrow Y(s) = \frac{s-1}{(s-2)^2(s+1)}$$

$$Y(s) = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

take laplace inverse

$$\mathcal{L}^{-1} Y(s) = \mathcal{L}^{-1}\left\{\frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}\right\}$$

$$= \frac{A(s-2)^2 + B(s+1)(s-2) + C(s+1)}{(s-2)^2(s+1)}$$

$$s - 1 = As^2 - 4As + 4A + Bs^2 - Bs - 2B + Cs + C$$

$$\left.\begin{array}{l} A + B = 0 \\ -4A - B + C = 1 \\ 4A - 2B + C = -1 \end{array}\right\} \Rightarrow A = -\frac{2}{9},\ B = \frac{2}{9},\ C = \frac{1}{3}$$

$$= \mathcal{L}^{-1}\left\{ \frac{-\frac{2}{9}}{S+1} + \frac{\frac{2}{9}}{S-2} + \frac{\frac{1}{3}}{(S-2)^2} \right\}$$

$$= y(t) = -\frac{2}{9} e^{-t} + \frac{2}{9} e^{2t} + \frac{1}{3} t e^{2t}$$

Ex $y'' - 5y' + 6y = 0$ Solve the ordinary diff equation using laplace trans. where $y(0) = 2$, $y'(0) = 2$

Solution

Take laplace: $\mathcal{L}\{y'' - 5y' + 6y\} = \mathcal{L}(0)$

$$S^2 Y(s) - 2S - 2 - 5[SY(s) - 2] + 6Y(s) = 0$$

$$Y(s)(S^2 - 5S + 6) = 2S - 8 \Rightarrow Y(s) = \frac{2S-8}{S^2-5S+6}$$

$$Y(s) = \frac{2S-8}{(S-3)(S-2)} = \frac{A}{S-3} + \frac{B}{S-2}$$

$$\mathcal{L}^{-1} Y(s) = \mathcal{L}^{-1}\left\{ \frac{A}{S-3} + \frac{B}{S-2} \right\}, \quad A = -2, \quad B = 4$$

$$y(t) = \mathcal{L}^{-1} \frac{-2}{S-3} + \mathcal{L}^{-1} \frac{4}{S-2}$$

$$y(t) = -2e^{3t} + 4e^{2t}$$

Ex Solve the (I.v.p.) $y''(t)+y(t)=3\sin 2t$
with $y(0)=1$, $y'(0)=-2$

Solution let $Y(s)=\mathcal{L}\{y(t)\}$

take laplace on both sides

$$\mathcal{L}y''(t)+\mathcal{L}y(t)=3\mathcal{L}\sin 2t$$

$$s^2Y(s)-sy(0)-y'(0)+Y(s)=\frac{6}{s^2+4} \Rightarrow s^2Y(s)-s+2+Y(s)=\frac{6}{s^2+4}$$

$$Y(s)(s^2+1)=\frac{s^3-2s^2+4s-2}{(s^2+4)} \Rightarrow Y(s)=\frac{s^3-2s^2+4s-2}{(s^2+4)(s^2+1)}$$

$$Y(s)=\frac{As+B}{(s^2+4)}+\frac{Cs+D}{(s^2+1)}=\frac{(As+B)(s^2+1)+(Cs+D)(s^2+4)}{(s^2+4)(s^2+1)}$$

$$s^3-2s^2+4s-2=As^3+As+Bs^2+B+Cs^3+4Cs+Ds^2+4D$$

$C=1$, $A=0$, $D=0$, $B=-2$

$$y(t)=\mathcal{L}^{-1}\frac{-2}{s^2+4}+\mathcal{L}^{-1}\frac{s}{s^2+1}$$

$A+C=1$
$B+D=-2$
$A+4C=4$
$B+4D=-2$

$$y(t)=-\sin 2t+\cos t$$

Ex Solve the intial value problem by laplace transform $y'' - y' - 2y = e^{2t}$ $y(0)=0,\ y'(0)=1$

Solu take laplace for both sides

$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - \mathcal{L}\{2y\} = \mathcal{L}\{e^{2t}\}$

$s^2 Y(s) - s y(0) - y'(0) - (sY(s) - y(0)) - 2Y(s)$

$s^2 Y(s) - 1 - sY(s) - 2Y(s) = \frac{1}{s-2}$

$Y(s)[s^2 - s - 2] = \frac{1}{s-2}$

$Y(s)[(s-2)(s+1)] = \frac{1}{s-2} + 1$

$= \frac{1+s-2}{(s-2)} = \frac{s-1}{(s-2)}$

$\therefore Y(s) = \frac{s-1}{(s-2)^2(s+1)} \Rightarrow Y(s) = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$

$= \frac{A(s-2)^2 + B(s+1)(s-2) + C(s+1)}{(s-2)^2(s+1)}$

$s - 1 = As^2 - 4As + 4A + Bs^2 - Bs - 2B + Cs + C$

$A + B = 0$ ---- (1) $\Rightarrow A = -B$ put in (2) (3)

$-4A - B + C = 1$ ---- (2) $\Rightarrow 4B - B + C = 1$

$4A - 2B + C = -1$ ---- (3) $\Rightarrow -4B - 2B + C = -1$

$$3B + C = 1$$

$$-6B + C = -1 \quad \text{طرح}$$

$$9B = 2 \Rightarrow \boxed{B = \frac{2}{9}} \qquad A = \frac{-2}{9}, \quad C = \frac{1}{3}$$

$$= \mathcal{L}^{-1} \frac{\frac{-2}{9}}{s+1} + \frac{\frac{2}{9}}{s-2} + \frac{\frac{1}{3}}{(s-2)^2}$$

$$y(t) = -\frac{2}{9} e^{-t} + \frac{2}{9} e^{2t} + \frac{1}{3} e^{2t} \mathcal{L}^{-1} \frac{1}{s^2}$$

$$y(t) = \frac{-2}{9} e^{-t} + \frac{2}{9} e^{2t} + \frac{1}{3} e^{2t} t$$

Ex $y'' - 5y' + 6y = 0 \qquad y(0) = 2$

$y'(0) = 2$

Solution

$$\mathcal{L}\{y'' - 5y' + 6y\} = \mathcal{L}\{0\}$$

$$s^2 Y_{(s)} - s\,y(0) - y'(0) - 5\{sY_{(s)} - y(0)\} + 6Y_{(s)} = 0$$

$$s^2 Y_{(s)} - 2s - 2 - 5sY_{(s)} + 10 + 6Y_{(s)} = 0$$

$$Y_{(s)}[s^2 - 5s + 6] = 2s - 8$$

$$Y_{(s)}[(s-3)(s-2)] = 2s - 8$$

$$Y_{(s)} = \frac{2s-8}{(s-3)(s-2)} = \frac{A}{s-3} + \frac{B}{s-2}$$

$$\mathcal{L}^{-1} Y_{(s)} = \mathcal{L}^{-1}\left\{\frac{A}{s-3} + \frac{B}{s-2}\right\}$$

$$A = \frac{2s-8}{\cancel{(s-3)}(s-2)}\cancel{(s-3)}\Big|_{s=3} = \frac{-2}{1} = \boxed{-2 = A}$$

$$B = \frac{2s-8}{(s-3)\cancel{(s-2)}}\cancel{(s-2)}\Big|_{s=2} = \frac{-4}{-1} = 4 = B$$

$$y(t) = \mathcal{L}^{-1} \frac{-2}{s-3} + \frac{4}{s-2} = -2e^{3t} + 4e^{2t}$$

The Solution of Linear System of Equations Numerically

Ageneral system of m linear equation with n unknown can be written as,

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

here $x_1, x_2, x_3, \cdots x_n$ are the unknowns
$a_{11}, a_{12}, \cdots, a_{mn}$ are the Coefficients of the system and $b_1, b_2, \cdots b_m$ are the Constant terms.

Matrix Equation

The matrix equation of the above linear system is

$$Ax = b$$

where A is an $(m \times n)$ matrix, x is a Column vector with n entries and b is a Column vector with m entries

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Asystem of linear eq. is homogenous if all the constant terms are zero ($b=0$)

$AX=0$

Numerical Methods

Numerical methods for solving linear system of eq. Can generally be divided into two Classes

* **Direct methods**: In the absence of round off error such methods would yield the exact solution within a finite number of steps such as LU method and Gaussian elimination method. and Gauss-Jordan method.
* **Iterative methods**: There are methods that are useful for problems involving special very large matrices, such as, Jacobi and Gauss. Seidel iterations method.

1- Gaussian Elimination Method

The goal of this method is to convert the Coefficients matrix "A" into an upper triangular matrix using forward elimination and then using back substitution to find "x_i", it is easy to illustrate this method by solving an example.

Ex Solve

$$2x_2 + x_4 = 0$$
$$2x_1 + 2x_2 + 3x_3 + 2x_4 = -2$$
$$4x_1 - 3x_2 + x_4 = -7$$
$$6x_1 + x_2 - 6x_3 - 5x_4 = 6$$

using Gauss elimination [four decimal places]

Solution:

* Form the matrix $(A \vdots b)$

$$(A \vdots b) = \left(\begin{array}{cccc|c} 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 6 & 1 & -6 & -5 & 6 \end{array}\right)$$

* We note that $a_{11} = 0$ and using Gauss-elimination we will have division by zero error B

to avoid this error we will use the "Partial pivo... ~~[illegible]~~, by rearrange the equations (rows), we put "a_{11}" the largest element regardless the sighn

$$\left(\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right)$$

* put $a_{i1}=0 \quad i=2,3,4$

using $R_i = R_i - (a_{i1}/a_{11}) R_1$

$$\left(\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 0 & 1.6667 & 5 & 3.6666 & -4 \\ 0 & -3.6666 & 4 & 4.3333 & -11 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right)$$

* Replacing R_3 instead of R_2:

$$\begin{pmatrix} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.6666 & 4 & 4.3333 & -11 \\ 0 & 1.6666 & 5 & 3.6666 & -4 \\ 0 & 2 & 0 & 1 & 0 \end{pmatrix}$$

* put $a_{i2}=0 \quad i=3,4$, $R_i = R_i - (a_{i2}/a_{22}) R_2$

$$\begin{pmatrix} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.6666 & 4 & 4.3333 & -11 \\ 0 & 0 & 6.8181 & 5.6362 & -8.9998 \\ 0 & 0 & 2.1818 & 3.3636 & -5.0001 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.6666 & 4 & 4.3333 & -11 \\ 0 & 0 & 6.8181 & 5.6362 & -8.9998 \\ 0 & 0 & 0 & 1.56 & \cancel{-2.12} \\ & & & & -3.1201 \end{array}\right)$$

This is anupper tringular matrix
now using back substitution to find x_4

$x_n = a_{n,n+1}/a_{nn} \Rightarrow x_4 = \frac{a_{45}}{a_{44}} = \Rightarrow (1.56\, x_4 = -2.12)$

$x_4 = \cancel{+1.3589}\ -2$

$x_i = \left(a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij} x_j\right)/a_{ii}$

to find $x_{n-1}, x_{n-2}, \dots x_1$

$x_3 = (a_{35} - a_{34} x_4)/a_{33} = \frac{-1.3407}{6.8181} = -0.1966$

$x_2 = [a_{25} - (a_{23} x_3 + a_{24} x_4)]/a_{22} = 1.1795$

$x_1 = [a_{15} - (a_{12} x_2 + a_{13} x_3 + a_{14} x_4)]/a_{11} = -0.5256$

Ex Solve for the value of x_1, x_2 and x_3 using Gaussian-elimination

$$-3x_1 + 2x_2 - x_3 = -1$$
$$6x_1 - 6x_2 + 7x_3 = -7$$
$$3x_1 - 4x_2 + 4x_3 = -6$$

(four decimal places)

Solution

* form the matrix $(A \vdots b)$

$$\begin{pmatrix} -3 & 2 & -1 & -1 \\ 6 & -6 & 7 & -7 \\ 3 & -4 & 4 & -6 \end{pmatrix} \Rightarrow \begin{array}{l} R_2 = R_2 - (\frac{6}{-3})R_1 \\ R_3 = R_3 - (-1)R_1 \end{array}$$

$$\begin{pmatrix} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & -2 & 3 & -7 \end{pmatrix} \Rightarrow R_3 = R_3 - R_2$$

$$\begin{pmatrix} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & 0 & -2 & 2 \end{pmatrix}$$

$x_3 = -1$

$x_2 = 2$

$x_1 = 2$

$$\left\{ \begin{array}{l} 2x - 6y - 2z = 5 \\ 8x + 3y - z = 2 \\ x + y + 4z = 4 \end{array} \right\}$$

H.W

Ex 1 Solve for the value of x_1, x_2 and x_3 using Gaussian-elimination method

$$-3x_1 + 2x_2 - x_3 = -1$$
$$6x_1 - 6x_2 + 7x_3 = -7 \qquad \text{(four decimal places)}$$
$$3x_1 - 4x_2 + 4x_3 = -6$$

Solution

Form the matrix $(A \vdots b)$

$$\left(\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 6 & -6 & 7 & -7 \\ 3 & -4 & 4 & -6 \end{array}\right)$$

* الغرض تحويل العناصر اسفل القطر الرئيسي للمصفوفة (A) الى "صفر" اي تكتب المعادلات الرياضية

Put $a_{i1} = 0 \qquad i = 2, 3, 4, 5, \dots \quad [i = 2, 3]$

using $R_i = R_i - (a_{i1}/a_{11}) R_1$

now! $i = 2 \Rightarrow a_{21} = 0$ using $R_2 = R_2 - (a_{21}/a_{11}) R_1$

$(a_{21}/a_{11}) = \frac{6}{-3} = \boxed{-2} \Rightarrow R_2 - (-2) R_1 \Rightarrow R_2 = R_2 + 2R_1$

$6 + 2(-3) = 0$

$-6 + 2(2) = -2$

$7 + 2(-1) = 5$

$-7 + 2(-1) = -9$

$$\begin{pmatrix} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ & & & \end{pmatrix}$$

now when $i=3$ $(a_{31}=0)$

$R_3 = R_3 - (a_{31}/a_{11}) R_1$

$= R_3 - (3/-3) R_1 = R_3 + R_1$

$3+(-3)=0$

$-4+2=-2$

$4+(-1)=3$

$-6+(-1)=-7$

$$\Rightarrow \left(\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & -2 & 3 & -7 \end{array}\right)$$

• Put $a_{i2}=0$ $\quad i=3,4,\dots$ $[i=3]$

using $R_i = R_i - (a_{i2}/a_{22}) R_2$

when $i=3 \Rightarrow R_3 = R_3 - (a_{32}/a_{22}) R_2$

$= R_3 - (-2/-2) R_2$

$R_3 = R_3 - R_2$

$0-0=0$

$-2-(-2)=0$

$3-5=-2$

$-7-(-9)=2$

$$\Rightarrow \left(\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & 0 & -2 & 2 \end{array}\right)$$

$$\begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -9 \\ 2 \end{pmatrix}$$

$\Rightarrow -3x_1 + 2x_2 - x_3 = -1 \qquad x_1 = 2$

$\Rightarrow 0x_1 - 2x_2 + 5x_3 = -9 \qquad x_2 = 2$

$\Rightarrow 0x_1 + 0x_2 - 2x_3 = 2 \Rightarrow x_3 = \frac{2}{-2} = -1$

Ex Solve

$$
\begin{aligned}
2x_2 + x_4 &= 0 \\
2x_1 + 2x_2 + 3x_3 + 2x_4 &= -2 \\
4x_1 - 3x_2 + x_4 &= -7 \\
6x_1 + x_2 - 6x_3 - 5x_4 &= 6
\end{aligned}
$$

using Gauss elimination (four decimal places)

Solution

* Form the matrix $(A \vdots b)$

$$
\left(\begin{array}{cccc|c}
0 & 2 & 0 & 1 & 0 \\
2 & 2 & 3 & 2 & -2 \\
4 & -3 & 0 & 1 & -7 \\
6 & 1 & -6 & -5 & 6
\end{array}\right)
$$

* we note $a_{11} = 0 \Rightarrow$ (division by zero error)
* to avoid this error we will use (The partial pivoting) by rearrange the equations (rows) we put "a_{ii}" the largest element regardless the sighn

$$
\left(\begin{array}{cccc|c}
6 & 1 & -6 & -5 & 6 \\
2 & 2 & 3 & 2 & -2 \\
4 & -3 & 0 & 1 & -7 \\
0 & 2 & 0 & 1 & 0
\end{array}\right)
$$

(1)

$$\left(\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right)$$

Put $a_{i1}=0 \quad i=2,3,4$

using : $R_i = R_i - (a_{i1}/a_{11}) R_1$

$i=2 \qquad (a_{21}=0)$

$(a_{21}/a_{11}) = \frac{2}{6} \Rightarrow R_2 = R_2 - \left(\frac{a_{21}}{a_{11}}\right) R_1$

$2 - \frac{2}{6}\, 6 = 0$

$2 - \frac{2}{6}(1) = \frac{10}{6} = 1.6666$

$3 - \frac{2}{6}(-6) = 5$

$2 - \frac{2}{6}(-5) = \frac{22}{6} = 3.6666$

$-2 - \frac{2}{6}(6) = -4$

$i=3 \qquad (a_{31}=0)$

$R_3 = R_3 - \frac{a_{31}}{a_{11}} R_1 \Rightarrow R_3 = R_3 - \frac{4}{6} R_1$

$4 - \frac{4}{6}\, 6 = 0$

$-3 - \frac{4}{6}(1) = -\frac{22}{6} = -3.6666$

$0 - \frac{4}{6}(-6) = 4$

$1 - \frac{4}{6}(-5) = \frac{26}{6} = 4.3333$

$-7 - \frac{4}{6}(6) = -11$

$$\left(\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 0 & \frac{10}{6} & 5 & \frac{22}{6} & -4 \\ 0 & -\frac{22}{6} & 4 & \frac{26}{6} & -11 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right)$$

using partial pivoting

$R_3 \Leftrightarrow R_2$

$$\left(\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 0 & -\frac{22}{6} & 4 & \frac{26}{6} & -11 \\ 0 & \frac{10}{6} & 5 & \frac{22}{6} & -4 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right)$$

$a_{i2} = 0 \quad i = 3, 4$

$R_i = R_i - (a_{i2}/a_{22}) R_2$

$a_{32}/a_{22} \Rightarrow R_3 = R_3 - \frac{a_{32}}{a_{22}}(R_2)$ $\left[\frac{a_{32}}{a_{22}} \quad \frac{\frac{10}{6}}{-\frac{22}{6}} = -\frac{10}{22}\right]$

$0 + \frac{10}{22} 0 = 0$ $\qquad R_3 - (-\frac{10}{22}) R_2 = R_3 + \frac{10}{22} R_2$

$\frac{10}{6} + \frac{10}{22}(-\frac{22}{6}) = 0$

$5 + \frac{10}{22}(4) = \frac{150}{22} \simeq 6.8181$

$\frac{22}{6} + \frac{10}{22}(\frac{26}{6}) = \frac{744}{132} = 5.6363 = \frac{372}{66} = \frac{186}{33} = \frac{62}{11}$

$-4 + \frac{10}{22}(-11) = -9$

$a_{42}/a_{22} \Rightarrow R_4 = R_4 - \frac{a_{42}}{a_{22}} R_2 \Rightarrow R_4 - (\frac{2}{-\frac{22}{6}}) R_2 = R_4 + \frac{12}{22} R_2$

$0 + \frac{12}{22} 0 = 0$

$2 + \frac{12}{22}(-\frac{22}{6}) = 0$

$0 + \frac{12}{22}(4) = \frac{48}{22} \simeq 2.1818$

$1 + \frac{12}{22}(\frac{26}{6}) = \frac{444}{132} \simeq 3.3636$

$0 + \frac{12}{22}(-11) = -6$

$$\left(\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 0 & -\frac{22}{6} & 4 & \frac{26}{6} & -11 \\ 0 & 0 & \frac{150}{22} & \frac{744}{132} & -9 \\ 0 & 0 & \frac{48}{22} & \frac{444}{132} & -6 \end{array}\right)$$

$\left(R_4 = R_4 - \frac{a_{43}}{a_{33}} R_3\right)$

(3)

$$\left(\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 0 & -\frac{22}{6} & 4 & \frac{26}{6} & -11 \\ 0 & 0 & \frac{150}{22} & \frac{744}{132} & -9 \\ 0 & 0 & \frac{48}{22} & \frac{444}{132} & -6 \end{array}\right)$$

$$R_4 = R_4 - \frac{\frac{48}{22}}{\frac{150}{22}} R_3 = R_4 - \frac{48}{\cancel{22}} \times \frac{\cancel{22}}{150} R_3 = \underline{R_4 - \frac{24}{75} R_3}$$

$$\frac{48}{22} - \frac{24}{75} \; \frac{150}{22} = 0$$

$$\frac{444}{132} - \frac{24}{75} \; \frac{744}{132} = \frac{15444}{9900} = 1.56$$

$$-6 - \frac{24}{75}(-9) = \frac{-450 + 216}{75} = \frac{-234}{75}$$

$$\left(\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 0 & -\frac{22}{6} & 4 & \frac{26}{6} & -11 \\ 0 & 0 & \frac{150}{22} & \frac{744}{132} & -9 \\ 0 & 0 & 0 & \frac{15444}{9900} & -\frac{234}{75} \end{array}\right)$$

This is an upper Tringular matrix now using back substitution to find X_4

$$X_n = a_{n,n+1} / a_{nn} \Rightarrow X_4 = \frac{a_{45}}{a_{44}}$$

to find X_i

$$X_i = \left(a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij} X_j \right) / a_{ii}$$

$$X_3 = (a_{35} - a_{34} X_4) / a_{33}$$

$$X_2 = [a_{25} - (a_{23} X_3 + a_{24} X_4)] / a_{22}$$

$$X_1 = [a_{15} - (a_{12} X_2 + a_{13} X_3 + a_{14} X_4)] / a_{11}$$

$$\begin{pmatrix} 6 & 1 & -6 & -5 \\ 0 & -\frac{22}{6} & 4 & \frac{26}{6} \\ 0 & 0 & \frac{150}{22} & \frac{744}{132} \\ 0 & 0 & 0 & \frac{15444}{9900} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 \\ -11 \\ -9 \\ -\frac{234}{75} \end{pmatrix}$$

$$0x_1 + 0x_2 + 0x_3 + \frac{15444}{9900}x_4 = -\frac{234}{75} \Rightarrow x_4 = \frac{-\frac{234}{75}}{\frac{15444}{9900}}$$

$$\boxed{\Rightarrow x_4 = -2}$$

$$\therefore 0x_1 + 0x_2 + \frac{150}{22}x_3 + \frac{744}{132}(-2) = -9 \Rightarrow \boxed{x_3 = \frac{1}{3}}$$

$$0x_1 - \frac{22}{6}x_2 + 4\left(\frac{1}{3}\right) + \frac{26}{6}(-2) = -11 \Rightarrow \boxed{x_2 = 1}$$

$$6x_1 + 1(1) - 6\left(\frac{1}{3}\right) - 5(-2) = 6 \Rightarrow \boxed{x_1 = -\frac{1}{2}}$$

Ex Solve: H.w

$x + 2y + z = 8$ $\quad (x=1,\ y=2,\ z=3)$

$3x + 4y + 2z = 17$

$6y - 20 = -5x - z$ $\quad$ (four decimal places)

Ex $4.44x - 9.94y + 2.41z = 5.36$

$8.24x + 2.02y - 2.18z = 9.34$

$0.93x + 3.86y + 11.66z = 2.57$

Solution

$$A:b = \left(\begin{array}{ccc|c} 4.44 & -9.94 & 2.41 & 5.36 \\ 8.24 & 2.02 & -2.18 & 9.34 \\ 0.93 & 3.86 & 11.66 & 2.57 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 8.24 & 2.02 & -2.18 & 9.34 \\ 4.44 & -9.94 & 2.41 & 5.36 \\ 0.93 & 3.86 & 11.66 & 2.57 \end{array}\right)$$

$$\begin{pmatrix} 8.24 & 2.02 & -2.18 & 9.34 \\ 0 & -11.0284 & 3.5846 & 0.3272 \\ 0 & 3.632 & 11.906 & 1.5158 \end{pmatrix} \Rightarrow \begin{pmatrix} 8.24 & 2.02 & -2.18 & 9 \\ 0 & -11.0284 & 3.5846 & 0.3 \\ 0 & 0 & 13.0865 & 1.623 \end{pmatrix}$$

$z = 0.124,\ y = 0.0296\cancel{4}7,\ x = 1.159$

Ex: Use Gaussian elimination to solve the system of linear eq.s (example)

$$2x_1 - 3x_2 + 4x_3 = 2$$
$$4x_1 + x_2 + 2x_3 = 2$$
$$x_1 - x_2 + 3x_3 = 3$$

Solution.

$$A|b \Rightarrow \left(\begin{array}{ccc|c} 2 & -3 & 4 & 2 \\ 4 & 1 & 2 & 2 \\ 1 & -1 & 3 & 3 \end{array}\right) R_1 \Longleftrightarrow R_3$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ 4 & 1 & 2 & 2 \\ 2 & -3 & 4 & 2 \end{array}\right) \begin{array}{l} R_2 = R_2 - 4R_1 \\ R_3 = R_3 - 2R_1 \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ 0 & 5 & -10 & -10 \\ 0 & -1 & -2 & -4 \end{array}\right)$$

$$R_2 = \frac{1}{5}R_2 \Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -2 & -4 \end{array}\right) R_3 = R_3 + R_2$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -4 & -6 \end{array}\right)$$

$$\therefore \quad x_3 = \frac{-6}{-4} = \frac{3}{2}, \quad x_2 = 1, \quad x_1 = \frac{-1}{2}$$

Example: Solve the system using Gaussian-elimination method:

$$2x_1+4x_2+4x_3+2x_4=16$$
$$4x_1+8x_2+6x_3+8x_4=32$$
$$14x_1+29x_2+32x_3+16x_4=112$$
$$10x_1+17x_2+10x_3+2x_4=28$$

Solution: form the matrix $(A:b)$

$$\left(\begin{array}{cccc|c} 2 & 4 & 4 & 2 & 16 \\ 4 & 8 & 6 & 8 & 32 \\ 14 & 29 & 32 & 16 & 112 \\ 10 & 17 & 10 & 2 & 28 \end{array}\right) \begin{array}{l} R_2=R_2-2R_1 \\ R_3=R_3-7R_1 \\ R_4=R_4-5R_1 \end{array}$$

$$\left(\begin{array}{cccc|c} 2 & 4 & 4 & 2 & 16 \\ 0 & \boxed{0} & -2 & 4 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & -3 & -10 & -8 & -52 \end{array}\right)$$

note: the main diagonal elements $\neq 0$.

So $R_2 \Leftrightarrow R_3$

$$\left(\begin{array}{cccc|c} 2 & 4 & 4 & 2 & 16 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & -2 & 4 & 0 \\ 0 & -3 & -10 & -8 & -52 \end{array}\right) R_4=R_4+3R_2$$

$$\left(\begin{array}{cccc|c} 2 & 4 & 4 & 2 & 16 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & -2 & 4 & 0 \\ 0 & 0 & 2 & -2 & -52 \end{array}\right) R_4=R_4+R_3 \left(\begin{array}{cccc|c} 2 & 4 & 4 & 2 & 16 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 2 & -52 \end{array}\right)$$

$\therefore x_4=-26,\ x_3=-52,\ x_2=260,\ x_1=-382$

Ex 2

Solve the System using Gaussian-elimination method

$$2x_1 - 3x_2 + 4x_3 = 2$$
$$4x_1 + x_2 + 2x_3 = 2$$
$$x_1 - x_2 + 3x_3 = 3$$

solution

$$(A \vdots b) \Rightarrow \left(\begin{array}{ccc|c} 2 & -3 & 4 & 2 \\ 4 & 1 & 2 & 2 \\ 1 & -1 & 3 & 3 \end{array}\right) R_1 \Leftrightarrow R_3$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ 4 & 1 & 2 & 2 \\ 2 & -3 & 4 & 2 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ 0 & 5 & -10 & -10 \\ 0 & -1 & -2 & -4 \end{array}\right)$$

$R_2 = R_2 - (\frac{4}{1})R_1$
$= R_2 - 4R_1$
$= 4 - 4(1) = 0$
$= 1 - 4(-1) = 5$
$= 2 - 4(3) = -10$
$= 2 - 4(3) = -10$

$R_3 = R_3 - (\frac{2}{1})R_1$
$= R_3 - 2R_1$
$= 2 - 2(1) = 0$
$= -3 - 2(-1) = -1$
$= 4 - 2(3) = -2$
$= 2 - 2(3) = -4$

$$\Rightarrow R_2 = \frac{1}{5}R_2 \Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -2 & -4 \end{array}\right)$$

$\Rightarrow R_3 = R_3 - (-1/1)R_2$
$R_3 = R_3 + R_2$

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -4 & -6 \end{array}\right)$$

$x_1 - x_2 + 3x_3 = 3$
$x_2 - 2x_3 = -2$
$\Rightarrow -4x_3 = -6$

$$x_3 = \frac{-6}{-4} = \frac{3}{2}, \quad x_2 = 1, \quad x_1 = -\frac{1}{2}$$

Iterative methods :-

These methods are usefulto Solve very larg[e] matrices

note : In these methods if we want t[o] reduce the iteration's Steps we must apply this condition:

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}| \qquad i = 1, 2, 3 \ldots n$$

Ex $n = 3$

$i = 1 \quad |a_{11}| \geq |a_{12}| + |a_{13}|$

$i = 2 \quad |a_{22}| \geq |a_{21}| + |a_{23}|$

$i = 3 \quad |a_{33}| \geq |a_{31}| + |a_{32}|$

$\vdots$

قيم معاملات العناصر القطرية في المنظومة الجبرية

مجموع معاملات باقي العناصر في نفس المعادلة

1- Jacobi method :

The system given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n &= b_n \end{aligned}$$

has a unique solution the coefficient matrix 'A' has no zeros on its main diagonal ($a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are non zeros)

Main ideas of Jacobi

* Rewritten the linear system of equations

$$x_1 = \frac{1}{a_{11}} (b_1 - a_{12} x_2 - a_{13} x_3 \; --- \; a_{1n} x_n)$$

$$x_2 = \frac{1}{a_{22}} (b_2 - a_{21} x_1 - a_{23} x_3 \cdots \; a_{2n} x_n)$$

$$\vdots$$

$$x_n = \frac{1}{a_{nn}} (b_n - a_{n1} x_1 - a_{n2} x_2 \; --- \; a_{nn-1} x_n)$$

* Make an intial guess of the solution
$x_1^{(0)} = x_2^{(0)} = x_3^{(0)} \cdots x_n^{(0)}$

* Substitute these values ito the right handside of the rewritten equations to obtian the first approximaton $(x_1^{(1)}, x_2^{(1)}, x_3^{(1)} \ldots x_n^{(1)})$ (one iteration) in the same way, the second, theird, -- nth iteration are computed.

* Continue iterations until two succesive approximation are identical

Ex (1) Solve using Jacobi method

$$-3x_1 + 9x_2 + x_3 = 2$$
$$5x_1 - 2x_2 + 3x_3 = -1$$
$$2x_1 - x_2 - 7x_3 = 3$$

Solution rearrange the equations such that
$|a_{ii}| > \sum |a_{ij}|$

$$5x_1 - 2x_2 + 3x_3 = -1$$
$$-3x_1 + 9x_2 + x_3 = 2$$
$$2x_1 - x_2 - 7x_3 = 3$$

re write the system; $x_1 = \frac{-1}{5} + \frac{2}{5}x_2 - \frac{3}{5}x_3$

$$x_2 = \frac{2}{9} + \frac{3}{9}x_1 - \frac{1}{9}x_3$$

$$x_3 = \frac{-3}{7} + \frac{2}{7}x_1 - \frac{1}{7}x_2$$

The intial value $x_1 = x_2 = x_3 = 0$

First approximation is

$$x_1^{(1)} = \frac{-1}{5} + \frac{2}{5}(0) - \frac{3}{5}(0) = \frac{-1}{5} = -0.2$$

$$x_2^{(1)} = \frac{2}{9} + \frac{3}{9}(0) - \frac{1}{9}(0) = 0.222$$

$$x_3^{(1)} = \frac{-3}{7} + \frac{2}{7}(0) - \frac{1}{7}(0) = -0.429$$

n	$K=0$ $x^{(0)}$	$K=1$ $x^{(1)}$	$K=2$ $x^{(2)}$	$K=3$ $x^{(3)}$	$K=4$ $x^{(4)}$	$K=5$ $x^{(5)}$	$K=6$ $x^{(6)}$	$x^{(7)}$
$x_1^{(k)}$	0	-0.2	0.146	0.192	0.181	0.185	0.186	0.186
$x_2^{(k)}$	0	0.222	0.203	0.328	0.332	0.329	0.331	0.331
$x_3^{(k)}$	0	$\bar{0}.429$	$\bar{0}.517$	$\bar{0}.416$	$\bar{0}.421$	$\bar{0}.424$	$\bar{0}.423$	$\bar{0}.423$

The Solutions is

$x_1 = 0.186$

$x_2 = 0.331$

$x_3 = -0.423$

Ex Solve using Jacobi method (Approximate the results into three decimal places)

$$-3x_1 + 9x_2 + x_3 = 2$$
$$5x_1 - 2x_2 + 3x_3 = -1$$
$$2x_1 - x_2 - 7x_3 = 3$$

Solution rearrange the system such that

$|a_{11}| \geq |a_{12}| + |a_{13}|$

$|a_{22}| \geq |a_{21}| + |a_{23}|$ $\quad |a_{ii}| \geq \sum |a_{ij}|$

$|a_{33}| \geq |a_{31}| + |a_{32}|$

$$5x_1 - 2x_2 + 3x_3 = -1 \quad \text{--- (1)}$$
$$-3x_1 + 9x_2 + x_3 = 2 \quad \text{--- (2)}$$
$$2x_1 - x_2 - 7x_3 = 3 \quad \text{--- (3)}$$

From (1) $5x_1 = -1 + 2x_2 - 3x_3 \Rightarrow \div 5$

$x_1 = (-1 + 2x_2 - 3x_3)/5 = \frac{-1}{5} + \frac{2}{5}x_2 - \frac{3}{5}x_3$

From (2) $9x_2 = 2 + 3x_1 - x_3 \Rightarrow \div 9$

$x_2 = (2 + 3x_1 - x_3)/9 = \frac{2}{9} + \frac{3}{9}x_1 - \frac{1}{9}x_3$

From (3) $-7x_3 = 3 - 2x_1 + x_2 \Rightarrow \div (-7)$

$x_3 = (3 - 2x_1 + x_2)/(-7)$

$x_3 = \frac{-3}{7} + \frac{2}{7}x_1 - \frac{1}{7}x_2$

$x_1 = (-1+2x_2-3x_3)/5 = -\frac{1}{5} + \frac{2}{5}x_2 - \frac{3}{5}x_3$

$x_2 = (2+3x_1-x_3)/9 = \frac{2}{9} + \frac{3}{9}x_1 - \frac{1}{9}x_3$

$x_3 = (3-2x_1+x_2)/(-7) = \frac{-3}{7} + \frac{2}{7}x_1 - \frac{1}{7}x_2$

let $x_1 = x_2 = x_3 = 0$ [Initial Value]

$\Rightarrow$ first approximation (itteration)

$x_1^{(1)} = (-1+2(0)-3(0))/5 = \frac{-1}{5} = -0.2$

or

$x_1^{(1)} = \frac{-1}{5} + \frac{2}{5}(0) - \frac{3}{5}(0) = -\frac{1}{5} = -0.2$

$x_2^{(1)} = (2+3(0)-0)/9 = \frac{2}{9} = 0.22222 \simeq 0.222$

or

$x_2^{(2)} = \frac{2}{9} + \frac{3}{9}(0) - \frac{1}{9}(0) = \frac{2}{9} \simeq 0.222$

$x_3^{(1)} = (3-2(0)+0)/(-7) = \frac{-3}{7} \simeq \bar{0}.42857 \simeq \bar{0}.429$

or

$x_3^{(1)} = \frac{-3}{7} + \frac{2}{7}(0) - \frac{1}{7}(0) = \frac{-3}{7} \simeq \bar{0}.429$

	K=0	K=1	
n	$x^{(0)}$	$x^{(1)}$	
$x_1^{(k)}$	0	$\bar{0}.2$	
$x_2^{(k)}$	0	0.222	
$x_3^{(k)}$	0	$\bar{0}.429$	

: Second itteration

$$x_1^{(2)} = (-1+2(0.222)-3(\bar{0}.429))/5 = \frac{0.731}{5} \simeq 0.1462 \simeq 0.146$$

$$x_2^{(2)} = (2+3(\bar{0}.2)-(0.\bar{4}29))/9 = \frac{1.829}{9} \simeq 0.20322 \simeq 0.203$$

$$x_3^{(2)} = (3-2(\bar{0}.2)+0.222)/(-7) = \frac{3.622}{6} \simeq \bar{0}.51742 \simeq \bar{0}.517$$

n	k=0 $x^{(0)}$	k=1 $x^{(1)}$	k=2 $x^{(2)}$
x_1	0	$\bar{0}.2$	0.146
x_2	0	0.222	0.203
x_3	0	$\bar{0}.429$	$\bar{0}.517$

$$x_1^{(3)} = (-1+2(0.203)-3(\bar{0}.517))/5 = \frac{0.957}{5} = 0.1914 \simeq 0.191$$

$$x_2^{(3)} = (2+3(0.146)-(-0.517))/9 = \frac{2.955}{9} \simeq 0.328$$

$$x_3^{(3)} = (3-2(0.146)+0.203)/(-7) = \frac{2.911}{-7} \simeq \bar{0}.41585 \simeq -0.416$$

n	$x^{(0)}$	$x^{(1)}$	$x^{(2)}$	$x^{(3)}$	$x^{(4)}$	$x^{(5)}$	$x^{(6)}$	$x^{(7)}$
x_1	0	$\bar{0}.2$	0.146	0.191	0.181	0.185	0.186	0.186
x_2	0	0.222	0.203	0.328	0.332	0.329	0.331	0.331
x_3	0	$\bar{0}.429$	$\bar{0}.517$	$\bar{0}.416$	$\bar{0}.421$	$\bar{0}.424$	$\bar{0}.423$	$\bar{0}.423$

∴ The Solution is

$x_1 = 0.186$, $x_2 = 0.331$, $x_3 = \bar{0}.423$

Ex (12) Solve using Jacobi method (5 iterations)

$$3x_2 - x_3 + 8x_4 = 15$$
$$-x_1 + 11x_2 - x_3 + 3x_4 = 25$$
$$10x_1 - x_2 + 2x_3 = 6$$
$$2x_1 - x_2 + 10x_3 - x_4 = -11$$

Solution: rearrange the equations:

$10x_1 - x_2 + 2x_3 = 6 \Rightarrow x_1 = (6 + x_2 - 2x_3)/10$

$-x_1 + 11x_2 - x_3 + 3x_4 = 25 \Rightarrow x_2 = (25 + x_1 + x_3 - 3x_4)/11$

$2x_1 - x_2 + 10x_3 - x_4 = -11 \Rightarrow x_3 = (-11 - 2x_1 + x_2 + x_4)/10$

$3x_2 - x_3 + 8x_4 = 15 \Rightarrow x_4 = (15 - 3x_2 + x_3)/8$

let $(x_1 = x_2 = x_3 = x_4 = 0) \Rightarrow$ The first approximate

Solution is

$x_1 = (6 + 0 - 2(0))/10 = 0.6$

$x_2 = (25 + 0 + 0 - 3(0))/11 = 2.2727$

$x_3 = (-11 - 0 + 0 + 0)/10 = -1.1$

$x_4 = (15 - 0 + 0)/8 = 1.875$

	$x^{(0)}$	$x^{(1)}$	$x^{(2)}$	$x^{(3)}$	$x^{(4)}$	$x^{(5)}$
x_1	0	0.6	1.0472	0.9326	1.0151	0.9889
x_2	0	2.2727	1.7159	2.0533	1.9536	2.0114
x_3	0	-1.1	$\overline{0}.8052$	$\overline{1}.0493$	$\overline{0}.9681$	$\overline{1}.0102$
x_4	0	1.875	0.8852	1.1308	0.9738	1.0213

(The exact solutions 1, 2, -1, 1)

التعليم العالي والبحث العلمي

مأمون الجامعة

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Numerical Methods To Solve Non Linear Equations

One of the most frequently occurring problems in Scientific work is to find the roots of equations of the form

$f(x) = 0$ ---- (1)

When both f and x are real numbers the value of x which satisfy the equation (1) is called the "root of equation" for example

$x^2 - 5x + 2 = 0$ ---- (2)

$x^3 + x^2 - 3x - 3 = 0$ ----(3)

$2^x - 5x + 2 = 0$ -------(4)

$e^x - 3x = 0$ ----- (5)

Eq. (2) is apolynomial of degree (2) which it's possible to obtain the exact roots

$x = \frac{5}{2} \mp \left[(\frac{5}{2})^2 - 2 \right]^{\frac{1}{2}}$

while it is difficalt [Sometimes impossible] to represent the roots in certain formula as in the rest of equations (3,4 and 5)

however we can hope to obtain only approxim
Solutions relying on some computational techn
-ues, So we will introduce some elementary
(numerical) iterative methods for finding
the solution of equation (1)

If $f(x_0)$· $[x_0, x_2]$ ea(5)
the new int
then
un

1. Bisection method
2. Newton's method
3. Secant method
4. Method of Linear interpolation [False Position]

Bisection method

Suppose the function $f(x)$ is continous on the interval $[x_0, x_1]$ such that $f(x_0)$ and $f(x_1)$ have opposite signs $[f(x_0) \cdot f(x_1) < 0]$, the difference signs of $f(x)$ in the points x_0, x_1 mea there at least one root (solution) for the equatio $f(x)=0$ in the interval $[x_0, x_1]$.

we halfe the interval (or bisect) s.th.

$x_2 = \frac{x_0 + x_1}{2}$, we get $[x_0, x_2]$ or $[x_2, x$

If $f(x_0) \cdot f(x_2) < 0$ then our new interval is $[x_0, x_2]$ eals,
the new interval is $[x_2, x_1]$ (it means $f(x_2) \cdot f(x_1) < 0$)
then we will continue subdividing (bisecting)
until $|x_i - x_{i-1}| < \varepsilon$ or $|f(x_i) - f(x_{i-1})| < \varepsilon$
(ε is avery small amount)

Bisection method can be slow, but it is simple and accurate. It is therefore sometimes used as a back up for more complicated methods.

Ex Bisect the interval $[1, 2]$ six times to find the root of the equation $f(x) = x^3 - x - 1$

Solution:

$x_1 = 1 \Rightarrow f(x_1) = -1$

$x_2 = 2 \Rightarrow f(x_2) = 5$

$x_3 = \frac{x_1 + x_2}{2} = \frac{1+2}{2} = 1.5$

to solve we will write the following table →

i	x_1	x_2	x_3	$f(x_1)$	$f(x_2)$	$f(x_3)$	
1	1	2	1.5	-1	5	0.875	$x_2 = x_3$
2	1	1.5	1.25	-1	0.875	$\bar{0}.2968$	$x_1 = x_3$
3	1.25	1.5	1.375	$\bar{0}.2968$	0.875	0.2246	$x_2 = x_3$
4	1.25	1.375	1.3125	$\bar{0}.2968$	0.2246	$\bar{0}.0515$	$x_1 = x_3$
5	1.3125	1.375	1.3437	$\bar{0}.0515$	0.2246	0.0823	$x_2 = x_3$
6	1.3125	1.3437	1.3281	$\bar{0}.0515$	0.0823	0.0144	$x_2 = x_3$

the root is $\dfrac{1.3125 + 1.3281}{2} = 1.3203$

and $f(1.3203) = \bar{0}.0187$

$f(x) = x + \ln x - 3 \quad [1, 3]$

i	x_1	x_2	x_3	$f(x_1)$	$f(x_2)$	$f(x_3)$	
1	1	3	2	-2	1.0986	$\bar{0}.3068$	$x_1 = x_3$
2	2	3	2.5	$\bar{0}.3068$	1.0986	0.4162	$x_2 = x_3$
3	2	2.5	2.25	$\bar{0}.3068$	0.4162	0.0609	$x_2 = x_3$
4	2	2.25	2.125	$\bar{0}.3068$	0.0609	$\bar{0}.1212$	$x_1 = x_3$
5	2.125	2.25	2.1875	$\bar{0}.1212$	0.0609	$\bar{0}.0297$	

Ex Bisect the interval [0.6, 0.8] five times to solve the eq. $2^x - 5x + 2 = 0$

Solution

$x_1 = 0.6 \qquad f(x_1) = 0.5157$

$x_2 = 0.8 \qquad f(x_2) = \bar{0}.2588$

$$x_3 = \frac{x_1 + x_2}{2} = \frac{0.6 + 0.8}{2} = 0.7$$

i	x_1	x_2	x_3	$f(x_1)$	$f(x_2)$	$f(x_3)$	
1	0.6	0.8	0.7	0.5157	$\bar{0}.2588$	0.1245	$x_1 = x_3$
2	0.7	0.8	0.75	0.1245	$\bar{0}.2588$	$\bar{0}.0682$	$x_2 = x_3$
3	0.7	0.75	0.725	0.1245	$\bar{0}.0682$	0.0279	$x_1 = x_3$
4	0.725	0.75	0.7375	0.0279	$\bar{0}.0682$	$\bar{0}.0202$	$x_2 = x_3$
5	0.725	0.7375	0.7312	0.0279	$\bar{0}.0202$	0.004	$x_1 = x_3$

So the root is $\dfrac{0.7375 + 0.7312}{2} = 0.7343$

and $f(0.7343) = \bar{0}.0079$

Ex

(1) $x - \sin x - 1 = 0$ $[1, 1.1]$

(2) $e^{x} - 3x = 0$ $[0.6, 0.7]$

(3) $x - \ln 3x = 0$ $[0.6, 0.7]$

(4) $x \log_{10} x - 1.2 = 0$ $[2.7, 2.8]$

note: We can determine an "x" interval containing a real root by:

* Compute (estimate) $f(x)$ for convenient values of "x" such as $0, \mp 1, \mp 2, \ldots$
* make rough sketch of $f(x)$

— $f(x) = \sin x - 2 + x$ $[0, 2]$

x_1	x_2	x_3	$f(x_1)$	$f(x_2)$	$f(x_3)$	
0	2	1	-2	0.9092	-0.1585	$x_1 = x_3$
1	2	1.5	-0.1585	0.9092	0.4974	$x_2 = x_3$
1	1.5	1.25	-0.1585	0.4974	0.1989	$x_2 = x_3$
1	1.25	1.125	-0.1585	0.1989	0.0272	$x_2 = x_3$
1	1.125	1.0625	-0.1585	0.02720	-0.0639	

Ex Bisect the interval $[1, 2]$ <u>Six times</u> to find the root (Solution) of the equation $f(x) = x^3 - x - 1$

<u>Solution</u>

$x_1 = 1 \Rightarrow f(x_1) = f(1) = (1)^3 - 1 - 1 = -1$

$x_2 = 2 \Rightarrow f(x_2) = f(2) = (2)^3 - 2 - 1 = 5$

$x_3 = \dfrac{x_1 + x_2}{2} = \dfrac{1+2}{2} = \dfrac{3}{2} = 1.5 \Rightarrow f(x_3) = f(1.5) = 0.875$

i	x_1	x_2	x_3	$f(x_1)$	$f(x_2)$	$f(x_3)$	
1	1	2	1.5	-1	5	0.875	$x_2 = x_3$
2	1	1.5	1.25	-1	0.875	$\bar{0}.2968$	$x_1 = x_3$
3	1.25	1.5	1.375	$\bar{0}.2968$	0.855	0.2246	$x_2 = x_3$
4	1.25	1.375	1.3125	$\bar{0}.2968$	0.2246	$\bar{0}.0515$	$x_1 = x_3$
5	1.3125	1.375	1.3437	$\bar{0}.0515$	0.2246	0.0823	$x_2 = x_3$
6	1.3125	1.3437	1.3281	$\bar{0}.0515$	0.0823	0.0144	

The root $= \underset{x_3}{1.3281}$

$f(\text{root}) = f(1.3281) = 0.0144$

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وزارة التعليم العالي والبحث العلمي

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Numerical Methods for Differantial Equation:

Numerical methods for differantial equations are of great importance to the engineer and physicist because practical problems often lead to differential equation that can not be solved exactly by the known method that solve differantial equations. Also there are Diff. Equations for which the solution in terms of formulas are so complecated that are prefers to apply a numerical method to such diff. Eq.

* Methods for first-order Diff. Eqs. :-

The general form of the first-order diff. equation is

$$F(x, y, y') = 0. \quad \text{---} (1)$$

and often it will be possible to write the equation in the explicit form

$$y' = f(x, y)$$

An initial value problem consists of a diff. equation and a condition the solution must satisfy. The initial value problem considered here is the form

$$y' = f(x,y) \quad , \quad y(x_0) = y_0$$

① Euler method

This method is the easiest method to solve the diff. Equation.

$$\boxed{y_{n+1} = y_n + h \cdot f(x_n, y_n)}$$

where $\boxed{x_{n+1} = x_n + h}$

and h is the step size.

Example Use Euler's method to obtain an approximate solution of Diff. Eq. solve to $x = 0.25$ where

$y' = x^2 + 4x - \frac{1}{2}y$ with $y(0) = 4$, $h = 0.05$

Sol. work to three digits (3D):

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

$h = 0.05$

$y_0 = 4$

$x_0 = 0$

$y' = f(x, y) = x^2 + 4x - \frac{1}{2}y$

$x_{n+1} = x_n + h$

$x_1 = x_0 + h = 0.05$

$x_2 = x_1 + h = 0.1$

$x_3 = 0.15$

$x_4 = 0.2$

$x_5 = 0.25$

for $n = 0$

$$y_1 = y_0 + h\left[x_n^2 + 4x_n - \tfrac{1}{2}y_n\right]$$

$$\therefore y_1 = y_0 + h\left[x_0^2 + 4x_0 - \tfrac{1}{2}y_0\right]$$

$$y_1 = 4 + 0.05\left[(0)^2 + 4(0) - \tfrac{1}{2}(4)\right]$$

$$y_1 = \boxed{3.9}$$

$n = 1$

$$y_2 = y_1 + h\left[x_1^2 + 4x_1 - \tfrac{1}{2}y_1\right]$$

$$= 3.9 + 0.05\left[(0.05)^2 + 4(0.05) - \tfrac{1}{2}(3.9)\right]$$

$$= \boxed{3.8}126$$

and so on ...

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X_n	y_n	Exact Solution [using the ordinary diff. Equation]
0	4	4
0.05	3.9	3.91
0.1	3.8126	3.82
0.15	3.7377	3.76
0.2	3.6753	3.7
0.25	3.62	3.65

Example Use Euler method for the following initial value problem, choosing $h=0.2$ and computing $y_1, \dots, y_5$

$$y' = xy \quad , \quad y(0) = 1 \; ?$$

Solution

$$y' = f(x,y) = xy$$

$$\Longrightarrow \quad f(x_n, y_n) = x_n \cdot y_n$$

$\therefore \quad y_0 = 1 \qquad h = 0.2$

$x_0 = 0$

$h = 0$

$y_{n+1} = y_n + h\, f(x_n, y_n)$ P3

for n = 0 $\quad y_1 = y_0 + h * (x_0 \cdot y_0)$

$y_1 = 1 + 0.2\,(0 * 1) \Rightarrow y_1 = 1$

n = 1 $\quad y_2 = y_1 + h\,(x_1 * y_1)$

$y_2 = 1 + 0.2\,(0.2 * 1) = 1.04$

n = 2

$y_3 = 1.04 + 0.2\,(0.4 * 1.04) = 1.1232$

n = 3

$y_4 = 1.1232 + 0.2\,(0.6 * 1.1232)$

$= 1.2580.$

$x_{n+1} = x_n + h$

$x_1 = x_0 + h$

$x_1 = 0 + 0.2$

$= 0.2$

$x_2 = x_1 + h$

$= 0.4$

$x_3 = 0.6$

$x_4 = 0.8$

$y_5 = 1.459.$

Home work

① solve the following diff. Eq. using Euler's method: $\quad y' = \frac{dy}{dx} = f(x,y) = \sqrt{x\sqrt{y}}$

to find $y(4)$ with $y(2) = 4$ & $h = 0.5$?

Example

$y' = f(x,y) = x^2 + 4x - \frac{1}{2}y$ find $x = 0.25 \Rightarrow y(0.25)$

$y(0) = 4$, $\cdot h = 0.05$

x_0 , y_0

$\Rightarrow x_0 = 0 \Rightarrow x_{n+1} = x_n + h$

$x_1 = x_0 + h = 0 + 0.05$

$x_1 = 0.05$

$x_2 = x_1 + h = 0.05 + 0.05 = 0.1 = x_2$

$x_3 = x_2 + h = 0.15$

$x_4 = 0.2$

$x_5 = 0.25$

	y_n	x_n	
y_0	4	0	x_0
y_1		0.05	x_1
y_2		0.1	x_2
y_3		0.15	x_3
y_4		0.2	x_4
y_5		0.25	x_5

$y_{n+1} = y_n + h f(x_n, y_n)$

$y_1 = y_0 + h f(x_0, y_0)$

$= 4 + 0.05 f(0, 4)$

$y_1 = 4 + 0.05\left(x^2 + 4x - \frac{1}{2}y\right)$

$y_1 = 4 + 0.05\left(0 + 4(0) - \frac{1}{2}(4)\right)$

$y_1 = 4 + 0.05(-2)$

$y_1 = 4 - 0.1 = 3.9$

$\boxed{y_1 = 3.9}$

	y_n	x_n	
y_0	4	0	x_0
y_1	3.9	0.05	x_1
y_2		0.1	x_2

$$y_2 = y_1 + h\,f(x_1, y_1)$$
$$y_2 = 3.9 + 0.05 f(0.05, 3.9)$$
$$y_2 = 3.9 + 0.05\left((0.05)^2 + 4(0.05) - \tfrac{1}{2}(3.9)\right)$$
$$y_2 = 3.9 + 0.05(-1.7475)$$
$$\boxed{y_2 = 3.8126}$$

	y_n	x_n	
y_0	4	0	x_0
y_1	3.9	0.05	x_1
y_2	3.8126	0.1	x_2
y_3		0.15	x_3

$$y_3 = y_2 + h f(x_2, y_2)$$
$$y_3 = 3.8126 + 0.05 f(0.1, 3.8126)$$
$$y_3 = 3.8126 + 0.05\left((0.1)^2 + 4(0.1) - \tfrac{1}{2}(3.8126)\right)$$
$$y_3 = 3.7377$$

	y_n	x_n	
y_0	4	0	x_0
y_1	3.9	0.05	x_1
y_2	3.8126	0.1	x_2
y_3	3.7377	0.15	x_3
y_4		0.2	x_4
y_5		0.25	x_5

$$y_4 = y_3 + h f(x_3, y_3)$$
$$= 3.7377 + 0.05 f(0.15, 3.7377)$$
$$= 3.7377 + 0.05\left((0.15)^2 + 4(0.15) - \tfrac{1}{2}(3.7377)\right)$$
$$\boxed{y_4 = 3.6753}$$

$$y_5 = y_4 + h f(x_4, y_4)$$
$$= 3.6753 + 0.05 f(0.2, 3.6753)$$
$$= 3.6753 + 0.05\left((0.2)^2 + 4(0.2) - \tfrac{1}{2}3.6753\right)$$
$$\boxed{y(0.25) = y_5 = 3.6254}$$

Ex Use Euler method for the following intial value problem, Choosing $h = 0.2$ and Computing $y_1, \ldots, y_5$

$y' = f(x, y) = xy$ with $y(0) = 1$

Solution

$h = 0.2$, $x_0 = 0$, $y_0 = 1$

find y_1, y_2, y_3, y_4, y_5

$y_{n+1} = y_n + h f(x_n, y_n)$

$y_1 = y_0 + h f(x_0, y_0)$

$= 1 + 0.2 f(0, 1)$

$= 1 + 0.2[xy] = 1 + 0.2[0(1)] = 1$

$\boxed{y_1 = 1}$

	x_n	y_n	
x_0	0	1	y_0
x_1	0.2		y_1
x_2	0.4		y_2
x_3	0.6		y_3
x_4	0.8		y_4
x_5	1		y_5

0.2

$y_2 = y_1 + h f(x_1, y_1)$

$= 1 + 0.2 f(0.2, 1)$

$= 1 + 0.2(0.2(1)) = 1 + 0.04$

$\boxed{y_2 = 1.04}$

	x_n	y_n	
x_0	0	1	y_0
x_1	0.2	1	y_1
x_2	0.4	1.04	y_2
x_3	0.6		y_3

$$y_3 = y_2 + hf(x_2, y_2)$$
$$= 1.04 + 0.2 f(0.4, 1.04)$$
$$= 1.04 + 0.2(0.4(1.04)) = \boxed{1.1232 = y_3}$$

	x_n	y_n	
x_0	0	1	y_0
x_1	0.2	1	y_1
x_2	0.4	1.04	y_2
x_3	0.6	1.1232	y_3
x_4	0.8	1.258	y_4
x_5	1		y_5

$$y_4 = y_3 + hf(x_3, y_3)$$
$$= 1.1232 + 0.2 f(0.6, 1.1232)$$
$$= 1.1232 + 0.2(0.6(1.1232)) = 1.25798 \simeq 1.258$$
$$\boxed{y_4 = 1.258}$$

$$y_5 = y_4 + hf(x_4, y_4)$$
$$= 1.258 + 0.2 f(0.8, 1.258)$$
$$= 1.258 + 0.2(0.8(1.258))$$
$$y_5 = 1.45928 \simeq \boxed{1.4593 = y_5}$$