Chapter 1

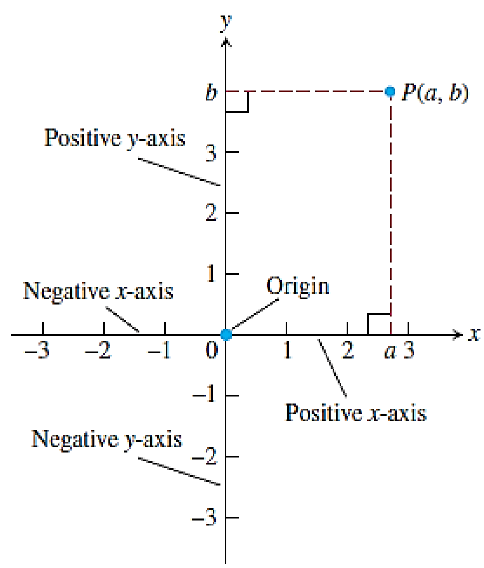
The rate of change of function.

المستوى الاحداثي

1.1 Cartesian coordinates in the plane

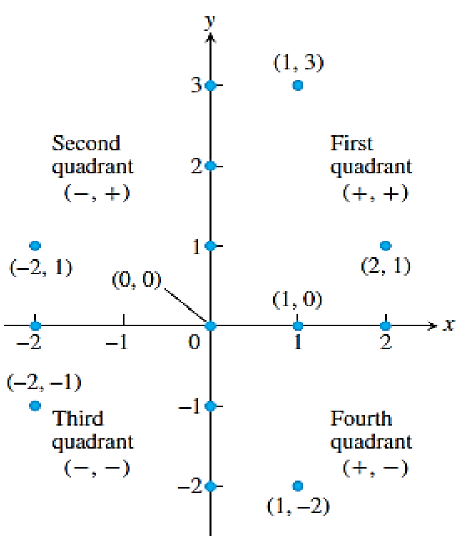
Points in the plane can be identified with ordered pairs of real number. To begin, we draw two perpendicular coordinate lines that intersect at the 0-point of each line.

عمودي



These lines are called Coordinate axes in the plane

احداثيات



The Coordinate axis of this coordinate or Cartesian plane divide the plane into four regions called "quadrant"

1.1.1 Increment and straight line:

التزايد (التغيير) والخط المستقيم

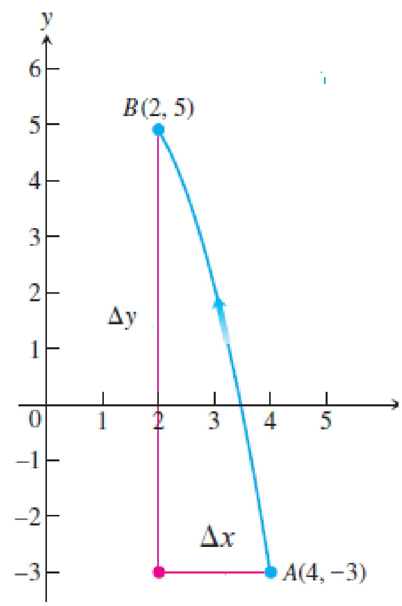
When a particle moves from one point in the plane to another, the net changes in its coordinates are called increments. They are calculated by subtracting the coordinates of the starting point from the coordinates of the ending point. If x changes from x_1 to x_2 the increment in x is $\Delta x = x_2 - x_1$

جسيم او شكل

EXAMPLE 1: A Particle moves from A to B in coordinate plane. Find the increment Δx and Δy in the particles coordinate.

Sol:

$$\Delta x = x_2 - x_1 \quad\quad y_2 = 5 \quad y_1 = -3$$
$$= 2 - 4 \quad\quad x_2 = 2 \quad x_1 = 4$$
$$= -2$$
$$\Delta y = y_2 - y_1$$
$$= 5 - (-3)$$
$$= 8$$



EXAMPLE 2: A Particle starts at A (-2, 3) and it's coordinates change by increments $\Delta x = 5$ and $\Delta y = -6$ find its new position?

النقطة الثانية

Sol:

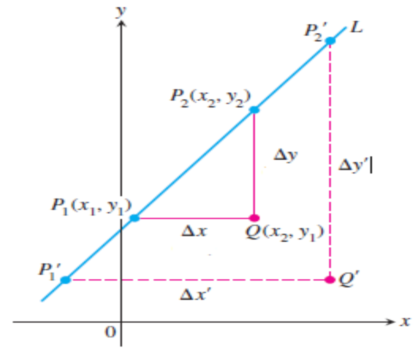
$$\Delta x = x_2 - x_1 \quad\quad \Delta y = y_2 - y_1$$
$$x_2 = \Delta x + x_1 \quad\quad y_2 = \Delta y + y_1$$
$$= 5 + (-2) \quad\quad = -6 + 3$$
$$= 3 \quad\quad = -3$$

The new position (3, -3)

1.2 The slop of straight line: ميل الخط المستقيم

$m=\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}$

Any non-vertical line in the plane has the property that the ratio $m=\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}$ has the same value for every choice of the two points and on the line.

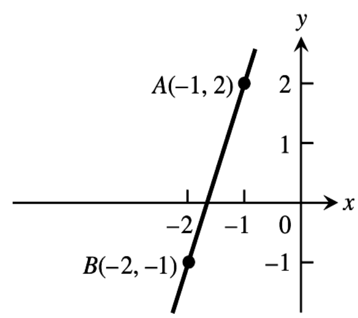


EXAMPLE 1 Plot the points and find the slope (if any) of the line they determine.

1) A (-1 , 2) B (-2 , -1)

2) A (2 , 3) B (-1 , 3)

Sol 1)

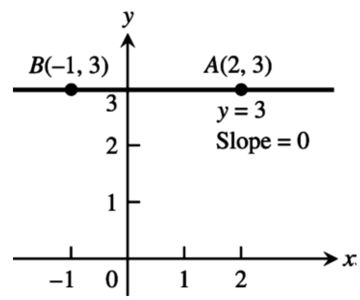


$$m=\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}$$

$$=\frac{-1-2}{-2+1}=\frac{-1}{-1}=3$$

2) $$m=\frac{\Delta y}{\Delta x}$$

$$m=\frac{3-3}{-1-2}=\frac{0}{-3}=0$$



EXAMPLE 2 Use slope to determine in each case whether the points are collinear (i.e. on a straight line) (هل توجد علاقة خطية)

1) A (1 , 0) B (0 , 1) C (2 , 1)

H.W A (-3 , -2) B (-2 , 0) C (-1 , 2) D (1 , 6)

mAB=mAC=mBC

Sol

$$m_{\text{AB}} = \frac{1-0}{0-1} = -1 \qquad m_{\text{AC}} = \frac{1-0}{2-1} = 1 \qquad m_{\text{BC}} = \frac{1-1}{2-0} = 0$$

$m_{\text{AB}} \neq m_{\text{AC}} \neq m_{\text{BC}}$ A, B and are no collinear

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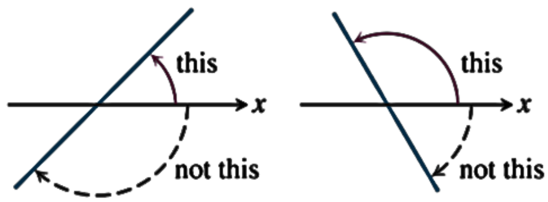
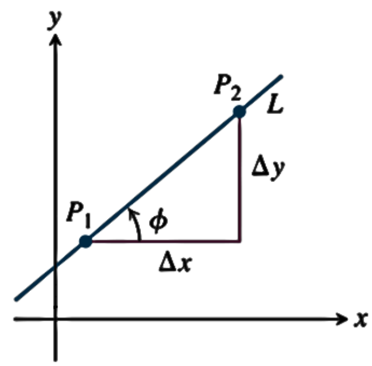
1.2 The angle of inclination



The relationship between the slope m of a nonvertical line and the line's angle of inclination ϕ is shown in Figure.

$$m = \frac{\Delta y}{\Delta x} = \tan\phi$$

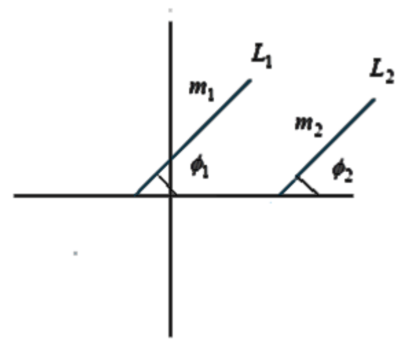


EXAMPLE 1 Find the slop of the line that make angle 60 with x- axis ?

Sol
$$\begin{aligned} m &= \tan\phi \\ &= \tan 60 \\ &= \sqrt{3} \end{aligned}$$

1.3.1 Parallel and perpendicular lines (توازي وتعامد الخطوط)

a) Lines that are parallel have equal angles of inclination, so they have the same slope (اذا كان المستقيمان متوازيان)



$L_1 // L_2 \quad , \quad \phi_1 = \phi_2$

$$m_1 = m_2$$

2. The slope of horizontal line is equal zero (ي هو صفر

b) If two lines are perpendicular, their slopes m_1 and m_2 satisfy (اذا كان الخطان متعامدان)

$$m_1 m_2 = -1$$

$$m_1 = \frac{-1}{m_2} \quad , \quad m_2 = \frac{-1}{m_1}$$

1.4 The slope form general equation (الميل من المعادلة العامة)

المعادلة العامة → $Ax + By = c$

$$m = \frac{-A}{B}$$

EXAMPLE 1 Find the slope of the line $2y = 3x + 4$

Sol

$2y = 3x + 4 \quad \rightarrow \quad A = -3 \,,\; B = 2$

$-3x + 2y = 4$

$$m = \frac{-A}{B}$$

$$m = \frac{-(-3)}{2} = \frac{3}{2}$$

EXAMPLE 2 Find the slope of the normal line $2x+3y=1$?

Sol

$2x+3y=1 \quad \Longrightarrow \quad A=2\ ,\ B=3$

$m=\frac{-2}{3}$

m' Perpendicular slope

$m'=-\frac{1}{m}=-\frac{1}{-\frac{2}{3}}=\frac{3}{2}$

EXAMPLE 3 Find the angle of inclination of the line $\sqrt{3}x+y=-3$?

Sol

$m=\frac{-\sqrt{3}}{1} \qquad -\sqrt{3}=\tan\vartheta$

$m=\tan\vartheta \qquad \theta=\tan^{-1}(-\sqrt{3}) \quad \Longrightarrow \quad =-60$

EXAMPLE 4 Find the slope of a line perpendicular to AB. A (1 , -2) B (2 , 1)

Sol

$m=\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}=\frac{1-(-2)}{2-1}=\frac{3}{1}$

$m=3$

$m'=-\frac{1}{m}=-\frac{1}{3}$ Perpendicular slope

معادلة الخط المستقيم

$y - y_1 = m\,(x - x_1)$

Equation of straight line:

a) Point –slope equation (معادلة نقطة وميل)

We can write an equation for a nonvertical straight line L if we know its slope m and the coordinates of one point on it.

$$m == \frac{y - y_1}{x - x_1}$$

$$y - y_1 = m\,(x - x_1)$$

$$\boxed{y = y_1 + m\,(x - x_1)}$$

The equation is the point - slope equation of the line that passes through the point (x_1, y_1) and has slope m.

EXAMPLE: Write an equation for line passes through (-1 , 1) with slope $m = -1$
Sol

$m = -1$ $\quad$ $y - y_1 = m\,(x - x_1)$ $\quad$ $x_1 = -1$, $y_1 = 1$

$$y - 1 = (-1)\,(x - (-1))$$

$$y - 1 = -x - 1$$

$$y = -x - 1 + 1$$

$$y = -x$$

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b) Slope intercept equation

A line with slope m and y-intercept b passes through the point (0, b), so it has equation.

$$\boxed{y = mx + b}$$

EXAMPLE: Write an equation for line has slope $-\frac{5}{4}$ and y-intercept 6.

Sol

$b = 6 \quad m = -\frac{5}{4}$ $\qquad b = 6$ y-intercept

$y = mx + b$

$y = -\frac{5}{4}x + 6$

EXAMPLE: Write an equation for line passes through (-1 , 1) and has slope 0.

Sol

$m = 0$ p (-1 , 1)

$y - y_1 = m\,(x - x_1)$

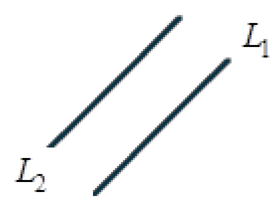
$y - 1 = 0\,(x - (-1))$

$y - 1 = 0$

$y = 1$

EXAMPLE : Write an equation for line passes through (5 ,- 1) and is parallel to the line $2x + 5y = 15$?.

Sol



$2x + 5y = 15$

$2x + 5y = 15 \longrightarrow L_1$

$m_1 = -2/5$ slope L_1

$L_1 // L_2 \qquad m_1 = m_2$

$m_2 = -2/5$ for L_2

We have $(-2/5)$ and point (5 ,- 1)

$$y - y_1 = m\,(x - x_1)$$

$$y - (-1) = -2/5\,(x - 5)$$

$$y + 1 = (-2/5)\,x + 2$$

$$y = (-2/5)\,x + 1$$

EXAMPLE : Write an equation for line passes through (4, 10) and perpendicular to the line $6x - 3y = 13$.

Sol

$$6x - 3y = 13 \qquad L_1$$

$$m_1 = -6/-3 = 2 \qquad L_1$$

m_2 (normal) (perpendicular)

$$m_2 = \frac{-1}{m_1}$$

$$m_2 = \frac{-1}{2} \text{ and } (4\ ,\ 10)$$

$$y - y_1 = m\,(x - x_1)$$

$$y - 10 = -\frac{1}{2}\,(x - 4)$$

$$y = -\frac{1}{2}x + 2 + 10$$

$$y = -\frac{1}{2}x + 12$$

Home work (H.W)

1) Find the slope of the line $\frac{x}{2} + \frac{y}{3} = 1$?

2) Find the equation of line that passes through P (2, -1) and parallel to the line $2x - 7y = 1$.

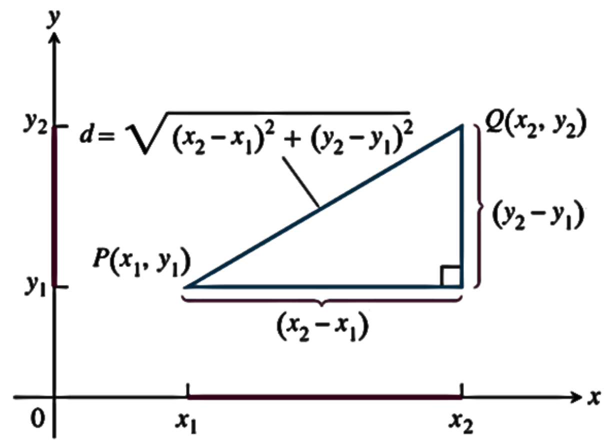
1.5 Distance between two points

المسافة بين نقطتين

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

The distance between points in the plane is calculated with a formula that comes from relation.

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$



EXAMPLE 1: Find distance between the point (1, -3) and (4, 2)

Sol

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

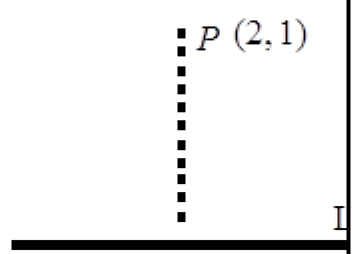
$$=\sqrt{(4-1)^2+(2-(-3))^2}$$

$$=\sqrt{9+25}$$

$$=\sqrt{34}$$

EXAMPLE 2: Find distance between the point P (2, 1) and the line L $y=x+2$?

Sol



$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

We have $y=x+2$ (1)

$m=\frac{-A}{B}$ $x-y=2$

$$m = \frac{-(-1)}{1} = 1$$

m' perpendicular $m = -\frac{1}{m}$

$m' = -1$

Equation of normal

$$y - y_1 = m\,(x - x_1)$$

$$y - 1 = -1\,(x - 2)$$

$$y - 1 = -x + 2$$

$$y + x = 3 \qquad (2)$$

لا يجاد المسافة بين نقطتين يجب ان نجد النقطة الثانية من خلال :

١. نجد ميل المستقيم الاول من المعادلة ونقلبه

٢. نجد معادلة المستقيم الثاني

٣. نكون معادلتين ونحلهم انيا ونجد النقطة

٤. نعوض النقطتين بقانون المسافة

$$y - x = 2$$

$$y + x = 3$$

$$2y = 5 \qquad y = 5/2$$

$$5/2 + x = 3 \qquad x = 1/2$$

Q (1/2, 5/2) , P (2, 1)

$$d = \sqrt{(1/2 - 2)^2 + (5/2 - 1)^2} = 2.12$$

H.W 3: Find distance between the point P (4, 6) and the line L: $4x + 3y = 12$?

Domain and Range:

1. The set D of all possible input values is call "Domain"
2. The set of all values of $f(x)$ as x varies throughout D is called Range of the function.
3. The domains D_f and range R_f of many functions in mathematics are interval of real number are shown in figure.

Root function

General form $y = \sqrt{f(x)} + k$

$D_f \longrightarrow f(x) \geq 0$

R_f we have two cases

الدومين هو جميع قيم x التي تجعل الدالة معرفة

الرنج هو النواتج المستخرجة عندما نعوض قيم x

الدالة الجذرية التربيعية تكون معرفة عندما لا يكون تحت الجذر قيمة سالبة

الدالة الكسرية تكون معرفة دائما عندما
لايوجد في المقام صفر

1. If the value of D_f unbounded

   $R_f = \{y: \; y \geq k\}$

Find domain D_f and range R_f of

EXAMPLE 1: $y = \sqrt{x}$

Sol

$x \geq 0$

$D_f = \{x : x \geq 0\}$

$R_f = \{y : y \geq 0\}$

EXAMPLE 2: $y = 1 + \sqrt{7 - 10x}$

Sol

$7 - 10x \geq 0$

$-10x \geq -7$

$10x \leq 7$

$x \leq 7/10$

$D_f = \{x : x \leq 7/10\}$

$R_f = \{y : y \geq 1\}$

H.W

Ex 3: $y = \sqrt{4 - x}$

Ex 4: $y = \sqrt{1 - 3x}$

Ex 5: $f(x) = \sqrt{2x - 3} + 7$

EXAMPLE3: $y = \sqrt{x^2 - 1}$

Sol

$x^2 - 1 \geq 0$

$x^2 \geq 1$

$x \ge \pm\sqrt{1}$

$x \ge \pm 1$

$x \ge 1 \;\text{ or }\; x \le -1$

$D_f = \{x : x \ge 1 \;\text{ or }\; x \le -1\}$

$R_f = \{y : y \ge 0\}$

$H.W:\; y = \sqrt{x^2 - 9}$

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2. If the value of D_f bounded $(-a \le x \le a)$

$R_f = (0 \le y \le a)$

EXAMPLE4: $y = \sqrt{4 - x^2}$

Sol

$4 - x^2 \ge 0$

$-x^2 \ge -4$

$x^2 \le 4$

$x \le \pm 2$

$D_f = \{x :\; -2 \le x \le 2\}$

$R_f = \{y :\; 0 \le y \le 2\}$ just positive

$H.W$ EX: $y = \sqrt{1 - x^2}$

$H.W$ Ex: $y = \sqrt{1 - \sqrt{x}}$

Ex: $y = \sqrt{7x - 3}$

Ex: $y = 1 - \sqrt{x}$

Relative function (الدالة العكسية)

General form $y = \dfrac{f(x)}{g(x)}$

$D_f = R / \{g(x) = 0\}$

To find R_f rewrite function ($x \to f(y)$)

Find domain D_f and range R_f

EXAMPLE 1: $y = \dfrac{1}{x}$

$D_f = R / \{x = 0\}$

$R_f \to \quad y = \dfrac{1}{x}$

$x = \dfrac{1}{y}$

$R_f = R / \{y = 0\}$

EXAMPLE 2: $y = \dfrac{3}{2x - 7}$

Sol

$2x - 7 = 0$

$2x = 7$

$x = 7 / 2$

$D_f = R / \{x = 7/2\}$

To find R_f

$y = \dfrac{3}{2x - 7}$

$2xy - 7y = 3$

$2xy = 3 + 7y$

$x = \dfrac{3 + 7y}{2y}$

$R_f = R / \{y = 0\}$

$H.W$ Ex 3: $y=\dfrac{x}{x-1}$

Ex 4: $y=\dfrac{3x-3}{1-7x}$

Ex 5: $y=\dfrac{2x-4}{x+3}$

EXAMPLE 6: $y=\dfrac{1}{\sqrt{7x-1}}$

Sol

$7x-1>0$

$7x>1$

$x>1/7$

$D_f=R/\{x:\ x>1/7\}$

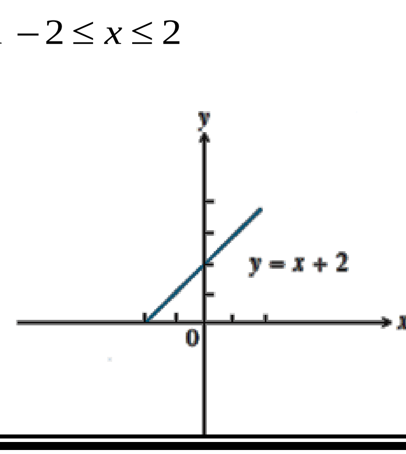
$R_f=R/\{y=0\}$

Graphs of functions (رسم الدوال)

Another way to visualize a function is its graph. If f is a function with domain D, its graph consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f.

EXAMPLE 1:Graph the function over the interval $-2\le x\le 2$

1. $y=x+2$



x	Y

2	4
1	3
0	2
-1	1
-2	0

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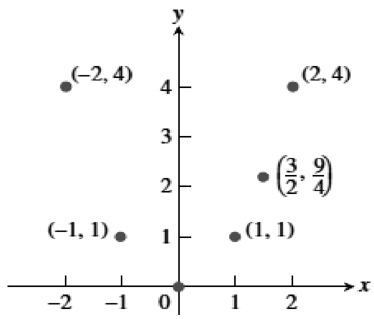
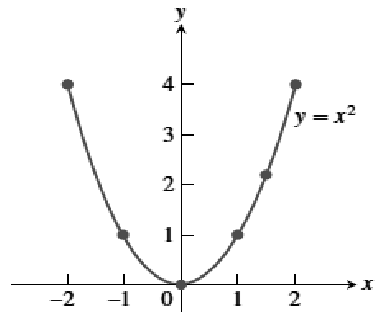
2. $y = x^2$





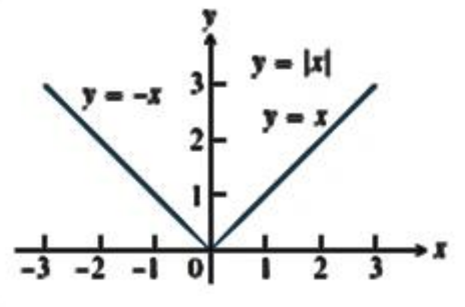
H.W Ex 2:Graph the function over the interval $-2 \le x \le 2$

1. $y = x^2 + 1$
2. $y = x^2 - 1$
3. $y = \sqrt{1 - x^2}$
4. $y = -\sqrt{1 - x^2}$

Graphing piecewise –Defined function

Sometimes a function is described by using different formulas on different part of its domain:

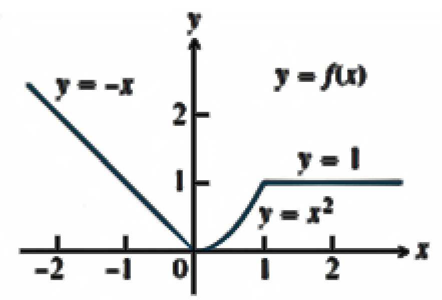
EXAMPLE 1: Graph the absolute value function



Sol

$$|x| = \begin{cases} x & x \ge 0 \\ -x & x < 0 \end{cases}$$

EXAMPLE 2: Graph the function $f(x) = \begin{cases} -x & x < 0 \\ x^2 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$



H.W Graph the functions $f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2 - x & 1 < x \le 2 \end{cases}$

10: Limits and continuity (الغاية والاستمرارية)

Let $f(x)$ be defined on an open interval about $x_\circ$, except possibly at $x_\circ$ itself. The limit of $f(x)$ as approaches $x_\circ$ is the number L $\lim\limits_{x \to x_\circ} f(x) = L$.

The limit of a function $f(x)$ as $x \to x_\circ$ never depend on what happen when $x = x_\circ$

Right hand limit $\lim\limits_{x \to x_\circ^+} f(x) = L$

Left hand limit $\lim\limits_{x \to x_\circ^-} f(x) = L$

A function $f(x)$ has a limit at point $x_\circ$ if and only if the right and left hand limit at $x_\circ$ exit and equal

$$\lim_{x \to x_\circ^+} f(x) = L \iff \lim_{x \to x_\circ^-} f(x) = L \quad \text{and} \lim_{x \to x_\circ} f(x) = L$$

10.1 The Limit Laws:

If L, M, C and K are real number and $\lim\limits_{x \to c} f(x) = L$, $\lim\limits_{x \to c} g(x) = M$

1) Sum rule $\lim\limits_{x \to c} (f(x) + g(x)) = L + M$
2) Deference $\lim\limits_{x \to c} (f(x) - g(x)) = L - M$
3) Product $\lim\limits_{x \to c} (f(x) \cdot g(x)) = L \cdot M$
4) Constant multiple $\lim\limits_{x \to c} (k \cdot f(x)) = k \cdot L$
5) Quotient rule $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, $M \neq 0$
6) power rule $\lim\limits_{x \to c} (f(x))^{r/s} = L^{r/s}$

EXAMPLE 1: Find the limit of the function $f(x) = x^3 + 4x^2 - 3$ at $x \to c$

Sol:

$$f(x) = x^3 + 4x^2 - 3$$

$$\lim_{x \to c} (x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3$$

$$= c^3 + 4c^2 - 3$$

H.W Ex 2: $\lim_{x \to c} \frac{x^4 - x^2 - 1}{x^2 + 5}$

H.W Ex 3: $\lim_{x \to 2} \sqrt{4x^2 - 3}$

Find the limit of function

a) The limit laws

1. $\lim_{x \to 2} (4)$
2. $\lim_{x \to 2} (5x - 3)$
3. $\lim_{x \to -2} \frac{3x + 4}{x + 5}$
4. $\lim_{x \to 1} (x^2 - 2x + 3)$ (To finding limits by calculating)
5. $\lim_{x \to 3} \sqrt{x^2 + 3x - 4}$
6. $\lim_{x \to -3} \frac{x^2 + x - 1}{2x - 4x^2}$
7. $\lim_{x \to 13} (4)$

b) Limit of Rational Function

EXAMPLE 1: $\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)}=\lim_{x\to 1}(x+1)=1+1=2$

Ex 2: $\lim_{x\to 2}\frac{x-2}{x^2-4}$

Ex 3: $\lim_{x\to -3}\frac{x^2-9}{x+3}$

Ex 4: $\lim_{x\to 2}\frac{x^4-16}{x-2}$

→ Limit of Rational function can be found by substitution if the limit of denominator is not zero

Ex 5: $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}$

Ex 6: $\lim_{x\to 4}\frac{x^2-3x-4}{x-4}$

Ex 7: $\lim_{x\to -4}\frac{x^2-16}{x^2-5x+4}$

Ex 8: $\lim_{x\to 5}\frac{x^2+5-6x}{x^2-25}$

Ex 9: $\lim_{x\to 5}\frac{x^3-8}{x^2-3x+2}$

Ex 10: $\lim_{x\to 5}\frac{x^4-16}{x^3-8}$

Hint $a^3-b^3=(a-b)(a^2+ab+b^2)$

$a^3+b^3=(a+b)(a^2-ab+b^2)$

Ex 1 $(x^3-8)=(x^3-2^3)=(x-2)(x^2+2x+4)$

c) Limit at infinity of Rational function

Hint

1) $\frac{\infty}{x} = \infty$

2) $\frac{x}{\infty} = 0$

3) $\frac{\infty}{0} = \infty$

To find Limit we can divide the numerator and denominator by highest power of x in denominator

If the value of $x \to \pm\infty$

EXAMPLE 1: $\lim_{x\to\infty} \frac{x^3 - 2x + 1}{2 + x^2 - 4x^3}$

Sol:

$$\lim_{x\to\infty} \frac{\frac{x^3}{x^3} - \frac{2x}{x^3} + \frac{1}{x^3}}{\frac{2}{x^3} + \frac{x^2}{x^3} - \frac{4x^3}{x^3}}$$

$$\lim_{x\to\infty} \frac{1 - \frac{2}{x^2} + \frac{1}{x^3}}{\frac{2}{x^3} + \frac{1}{x} - 4} = \frac{1 - \frac{2}{\infty} + \frac{1}{\infty}}{\frac{2}{\infty} + \frac{1}{\infty} - 4} = \frac{1 - 0 + 0}{0 + 0 - 4}$$

$$\lim_{x\to\infty} \frac{x^3 - 2x + 1}{2 + x^2 - 4x^3} = -\frac{1}{4}$$

Ex 2: $\lim_{x\to\infty} \frac{4x^5 + 2x^3 + 3x - 1}{2x^7 - 3x^4 + 2x - 7}$ ans (0)

Ex 3: $\lim_{x\to\infty} \frac{2x - 3}{x + \sqrt{x^2 - 1}}$ ans (1)

Ex 4: $\lim_{x\to\infty} \frac{7x + 2}{3x + \sqrt{x^2 + 1}}$ ans (7/4)

d) Limit of Root function

If we have Root $\pm$ number

Root $\pm$ Root

EXAMPLE 1: Find limit of $\frac{1-\sqrt{x+1}}{x}$ at $x \to 0$

Sol:

$$\lim_{x\to 0}\frac{1-\sqrt{x+1}}{x}\cdot\frac{1+\sqrt{x+1}}{1+\sqrt{x+1}}=\lim_{x\to 0}\frac{1-(x+1)}{x+x\sqrt{x+1}}$$

$$=\lim_{x\to 0}\frac{1-x-1}{x(1+\sqrt{x+1})}=\lim_{x\to 0}\frac{-x}{x(1+\sqrt{x+1})}=\frac{-1}{(1+\sqrt{x+1})}=\frac{-1}{(1+\sqrt{0+1})}$$

$$=\frac{-1}{(1+1)}=-\frac{1}{2}$$

H.W Ex 2 $\lim_{n\to\infty}\sqrt{n^2+1}-n$

EXAMPLE 3: $\lim_{x\to 1}\frac{\sqrt{x+1}-\sqrt{2x}}{x^2-x}$

Sol:

$$\lim_{x\to 1}\frac{\sqrt{x+1}-\sqrt{2x}}{x^2-x}=\lim_{x\to 1}\frac{\sqrt{x+1}-\sqrt{2x}}{x^2-x}\cdot\frac{\sqrt{x+1}+\sqrt{2x}}{\sqrt{x+1}+\sqrt{2x}}$$

$$\lim_{x\to 1}\frac{x+1-2x}{(x^2-x)(\sqrt{x+1}+\sqrt{2x})}=\lim_{x\to 1}\frac{1-x}{x\,(x-1)(\sqrt{x+1}+\sqrt{2x})}$$

$$=\lim_{x\to 1}\frac{-(x-1)}{x\,(x-1)(\sqrt{x+1}+\sqrt{2x})}=\lim_{x\to 1}\frac{-1}{x(\sqrt{x+1}+\sqrt{2x})}$$

$$=\lim_{x\to 1}\frac{-1}{1\cdot(\sqrt{1+1}+\sqrt{2\cdot 1})}=-\frac{1}{2\sqrt{2}}$$

H.W Ex 4: $\lim_{x\to 0}\frac{5x^3+8x^2}{3x^4-16x^2}$

H.W Ex 5: $\lim_{x\to 0}\frac{x^3-a^3}{x^4-a^4}$

11. Continuity (الاستمرارية)

Continuous function: A function is continuous if it is continuous at each point of its domain.

The Continuity test

The function $y = f(x)$ is continuous at $x = c$ if and only if all three of following statement are true

1. $f(c)$ exit "c in the domain of f"
2. $\lim_{x\to c} f(x)$ exit "f has a limit at $x \to c$"
3. $\lim_{x\to c} f(x) = f(c)$ "The limit equal the function value"

Hint if f continues at $x \to c$ and g continuous at $x \to c$

1. $f \cdot g$
2. $g \cdot f$
3. $k \cdot g$
4. f / g

.... continuous

EXAMPLE 1: Determine if the following function is continuous at $x = 1$?

$$f(x) = \begin{cases} 3x - 5 & \text{at } x \neq 1 \\ 2 & \text{at } x = 1 \end{cases}$$

Sol:

1) $f(1) = 2$
2) $\lim_{x\to 1} f(x) = \lim_{x\to 1}(3x - 5)$

$= (3(1) - 5) = -2$

$f(1) \neq \lim_{x\to 1} f(x)$

The function $f(x)$ is not continuous at $x = 1$

H.W **Ex 2**: Determine if the $f(x)$ is continuous at $x=3$?

$$f(x)=\begin{cases} x^2-1 & \text{at } x=3 \\ x^2+2 & \text{at } x\neq 3 \end{cases}$$

EXAMPLE 3: if $f(x)=\begin{cases} \dfrac{x^2-1}{x-2} & \text{at } x>2 \\ x+2 & \text{at } x\leq 2 \end{cases}$ continuous at $x=2$?

Sol:

1) $f(x)=x+2$ at $x=2$

$f(2)=2+2=4$

2) $\lim\limits_{x\to 2} f(x)=\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}=\lim\limits_{x\to 2}\dfrac{(x-2)(x+2)}{x-2}$

$\lim\limits_{x\to 2}(x+2)=2+2=4$

$f(2)=\lim\limits_{x\to 2}(x+2)$

$f(x)$ is continuous

H.W **Ex 4**: if $f(x)=\dfrac{x+3}{x^2-1}$ where is $f(x)$ continuous, and where it is discontinuous ?

EXAMPLE 5: if $f(x)=\begin{cases} x^2+2ax+3 & x=2 \\ x+4 & x\neq 2 \end{cases}$ find the value a if $f(x)$ continuous.

Sol:

1) $f(x)=x^2+2a\,x+3$

$f(2)=2^2+2a(2)+3$

$f(2)=7+4a$

2) $\lim_{x\to 2}(x+4)=(2+4)=6$

If $f(x)$ is continuous

$$f(2)=\lim_{x\to 2}(x+4)$$

$$7+4a=6$$

$$4a=-1 \quad \Rightarrow \quad a=-1/4$$

---

limit of trigonometric function (غاية الدوال المثلثية)

1. $\lim_{x\to 0}\sin x=0$ $\qquad \sin(0)=0$
2. $\lim_{x\to 0}\cos x=1$ $\qquad \cos(0)=1$
3. $\lim_{x\to 0}\tan x=0$ $\qquad \tan(0)=0$
4. $\lim_{x\to 0}\frac{\sin x}{x}=1$
5. $\lim_{x\to 0}\frac{\sin ax}{ax}=\lim_{x\to 0}\frac{ax}{\sin ax}=1$
6. $\lim_{x\to 0}\frac{\tan ax}{ax}=\lim_{x\to 0}\frac{ax}{\tan ax}=1$
7. $\lim_{x\to 0}\frac{1-\cos x}{x}=0$

EXAMPLE 1: Prove $\lim_{x\to 0}\sin x=0$

Sol: $\lim_{x\to 0}\sin x=\sin(0)=0$

EXAMPLE 2: Prove $\lim_{x\to 0}\cos x=1$

Sol: $\lim_{x\to 0}\cos x=\cos(0)=1$

EXAMPLE 3: Prove $\lim_{x\to 0}\tan x=0$

Sol: $\lim_{x\to 0}\tan x=\tan(0)=0$

EXAMPLE 6: Find the limit the following function at $x \to 0$

1) $\frac{\sin 5x}{x}$ 2) $\frac{\sin x}{7x}$ 3) $\frac{\tan 3x}{x}$ 4) $\frac{\sin 5x}{7x}$ 5) $\frac{2x}{4\tan(7x)}$

6) $\frac{\sin 2x}{2x^2 + x}$ 7) $\frac{\sin 3x - \tan 2x}{5x}$ 8) $\frac{\sin 7x}{\tan 3x}$ 9) $\frac{\tan^2 x}{x \sin 2x}$

10) $\sin(\pi / 2\cos(\tan x))$

1) Sol:

$$= \lim_{x\to 0} \frac{\sin 5x}{x} = \lim_{x\to 0} \frac{\sin 5x}{x} \frac{5}{5}$$

$$= 5 \lim_{x\to 0} \frac{\sin 5x}{5x}$$

$$= 5\,(1) = 5$$

3) Sol:

$$= \lim_{x\to 0} \frac{\tan 3x}{x} = \lim_{x\to 0} \frac{\tan 3x}{x} \frac{3}{3}$$

$$= 3 \lim_{x\to 0} \frac{\tan 3x}{3x}$$

$$= 3\,(1) = 3$$

5) Sol:

$$= \lim_{x\to 0} \frac{2x}{4\tan(7x)} = \frac{7}{7} \lim_{x\to 0} \frac{2x}{4\tan(7x)}$$

$$= \frac{2}{4}\frac{1}{7} \lim_{x\to 0} \frac{7x}{\tan(7x)}$$

$$= \frac{2}{28}(1)$$

$$= \frac{1}{14}$$

6) Sol:

$$= \lim_{x\to 0} \frac{\sin 2x}{2x^2 + x} = \lim_{x\to 0} \frac{\sin 2x}{x(2x+2)} = \lim_{x\to 0} \frac{\sin 2x}{x} \cdot \lim_{x\to 0} \frac{1}{2x+1}$$

$$= \lim_{x\to 0}\frac{\sin 2x}{x}\cdot \lim_{x\to 0}\frac{1}{2x+1}$$

$$= \frac{2}{2}\lim_{x\to 0}\frac{\sin 2x}{x}\cdot \lim_{x\to 0}\frac{1}{2x+1}$$

$$= 2\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot \lim_{x\to 0}\frac{1}{2x+1}$$

$$= 2(1)\cdot(\frac{1}{0+1})$$

$$= 2$$

7) Sol:

$$= \lim_{x\to 0}\frac{\sin 3x-\tan 2x}{5x} = \lim_{x\to 0}\frac{\sin 3x}{5x} - \lim_{x\to 0}\frac{\tan 2x}{5x}$$

$$= \frac{3}{3}\lim_{x\to 0}\frac{\sin 3x}{5x} - \frac{2}{2}\lim_{x\to 0}\frac{\tan 2x}{5x}$$

$$= \frac{3}{5}\lim_{x\to 0}\frac{\sin 3x}{3x} - \frac{2}{5}\lim_{x\to 0}\frac{\tan 2x}{2x}$$

$$= \frac{3}{5}(1) - \frac{2}{5}(1) = \frac{1}{5}$$

9) Sol:

$$= \lim_{x\to 0}\frac{\tan^2 x}{x\sin 2x} = \lim_{x\to 0}\frac{\left(\dfrac{\tan x}{x}\right)^2}{x\,\dfrac{2}{2}\,\dfrac{\sin 2x}{x^2}} = \lim_{x\to 0}\frac{\left(\dfrac{\tan x}{x}\right)^2}{x\,\dfrac{2\sin 2x}{2x^2}}$$

$$= \lim_{x\to 0}\frac{\left(\dfrac{\tan x}{x}\right)^2}{2\cdot\dfrac{\sin 2x}{2x}} = \frac{1}{2}$$

Exercises

Q1) Find the limits

a) $= \lim_{x \to -1} \frac{3x^2}{2x-1}$

b) $= \lim_{x \to \pi/2} x \sin x$

c) $= \lim_{x \to \pi} \frac{\cos x}{1-\pi}$

Q2) Calculate limits using the limit laws

a) $= \lim_{t \to 1} \frac{t^2 + 3t + 2}{t^2 - t - 2}$

b) $= \lim_{x \to -1} \frac{\sqrt{x^2+8} - 3}{x+1}$

c) $= \lim_{u \to 1} \frac{u^4 - 1}{u^3 - 1}$

d) $= \lim_{x \to 0} \frac{\sqrt{x^2+100} - 10}{x^2}$

e) $= \lim_{x \to 4} \frac{4x - x^2}{2 - \sqrt{x}}$

Q3) Using $= \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

Show that a) $= \lim_{h \to 0} \frac{\cos h - 1}{h} = 0$

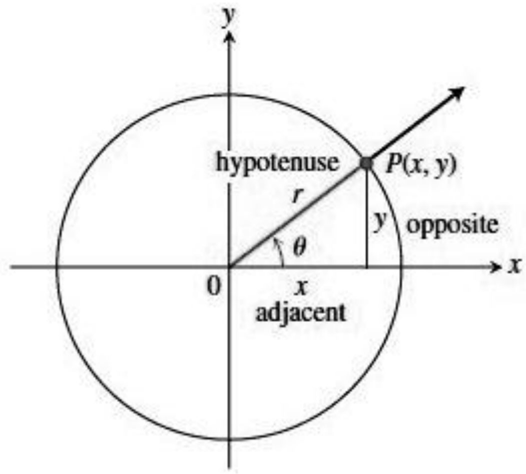
b) $= \lim_{x \to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$

Trigonometry function (الدوال المثلثية)

Basic Algebra, Geometry and Trigonometry formulas

Trigonometry formulas

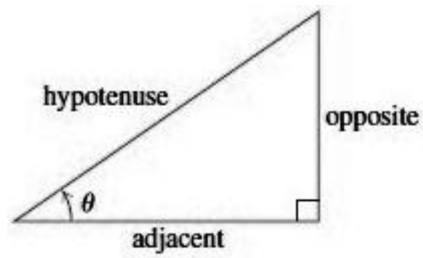
Definitions and fundamental identities



Sine $$\sin\theta = \frac{y}{r} = \frac{1}{\csc\theta}$$

Cosine $$\cos\theta = \frac{x}{r} = \frac{1}{\sec\theta}$$

Tangent $$\tan\theta = \frac{y}{x} = \frac{1}{\cot\theta}$$



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	**−180**	**−135**	**−90**	**−45**	**0**	**30**	**45**	**60**	**90**	**120**	**135**	**150**	**180**	**270**	**360**
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos\theta$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
$\tan\theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

The Basic Trigonometry function (قوانين الدوال المثلثية)

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} =$ $\cot\theta = \dfrac{1}{\tan\theta}$ $\sec\theta = \dfrac{1}{\cos\theta} =$ $\csc\theta = \dfrac{1}{\sin\theta}$

Even	odd
$\cos(-x) = \cos(x)$	$\sin(-x) = -\sin(x)$
$\sec(-x) = \sec(x)$	$\tan(-x) = -\tan(x)$
	$\csc(-x) = -\csc(x)$
	$\cot(-x) = -\cot(x)$

$$\boxed{\sin^2\theta + \cos^2\theta = 1} \qquad (1)$$

This equation, true for all values of θ is the most frequently used identity in trigonometry. Dividing this identity in turn by $\cos^2\theta$ and $\sin^2\theta$ gives

$$\boxed{\begin{aligned}1+\tan^2\theta &= \sec^2\theta\\ 1+\cot^2\theta &= \csc^2\theta\end{aligned}}$$

Addition formulas

$$\boxed{\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B\\ \sin(A+B) &= \sin A\cos B + \cos A\sin B\end{aligned}} \qquad (2)$$

Double –Angle Formulas

$$\boxed{\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ \sin 2\theta &= 2\sin\theta\cos\theta\end{aligned}} \qquad (3)$$

Additional formulas come from combining the equations $\sin^2\theta + \cos^2\theta = 1$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$ we add the two equations to get $2\cos^2\theta = 1 + \cos 2\theta$ and subtract the second from the first to get $2\sin^2\theta = 1 - \cos 2\theta$. This results in the following identities, which are useful in integral calculus.

$$\boxed{\cos^2\theta = \frac{1+\cos 2\theta}{2}} \qquad (4)$$

$$\boxed{\sin^2\theta = \frac{1-\cos 2\theta}{2}} \qquad (5)$$

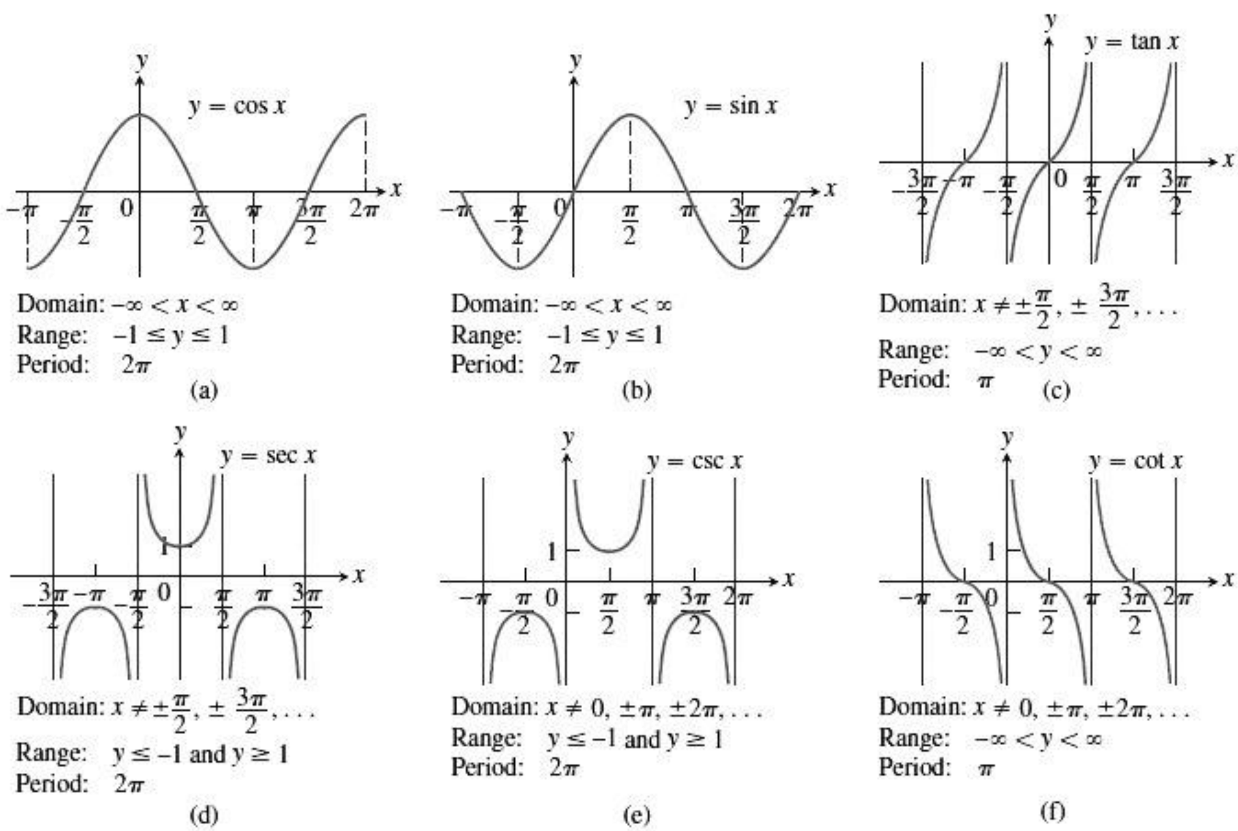
DEFINITION Periodic Function

A function $f(x)$ is periodic if there is a positive number p such that $f(x+p)$ for every value of x. The smallest such value of p is the period of f.

Periodic Trigonometric function

Periodic π
$$\tan(x+\pi) = \tan x$$
$$\cot(x+\pi) = \cot x$$

Periodic 2π
$$\sin(x+2\pi) = \sin x$$
$$\cos(x+2\pi) = \cos x$$
$$\sec(x+2\pi) = \sec x$$
$$\csc(x+2\pi) = \csc x$$



Graphs of the (a) cosine, (b) sine, (c) tangent, (d) secant, (e) cosecant, and (f) cotangent functions using radian measure. The shading for each trigonometric function indicates its periodicity.

<u>Law of exponents</u>

$$a^m a^n = a^{m+n} \qquad (ab)^m = a^m b^m \qquad (a^m)^n = a^{mn} \qquad a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

If $a \neq 0$,

$$\frac{a^m}{a^n} = a^{m-n} , \quad a^0 = 1 \qquad a^{-m} = \frac{1}{a^m}$$

The Binomial Theorem for any positive integer n

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots nab^{n-1} + b^n$$

For instant

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad (a-b)^2 = a^2 - 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \qquad (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

Transcendental function

1.1 Logarithm function

- Definition $\log_a x$
- Properties of Logarithm function
- Rule of Logarithm function
- Example

1.2 Exponential function

- Definition of
- Properties and rule of Exponential function
- Example

1.3 Invers function

- Example

1.1 Logarithm function

Logarithms with Base a

Definition $\log_a x$

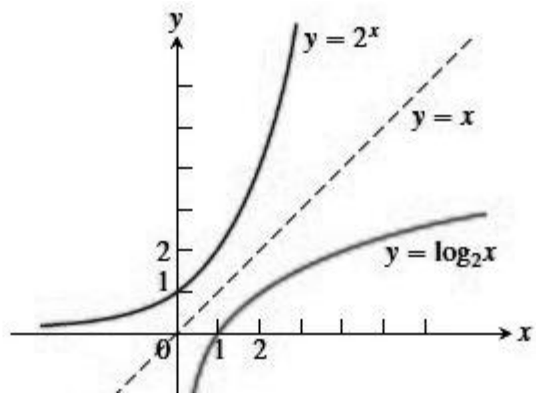
For any positive number $a \neq 1$ $\log_a x$ is the inverse function of a^x .

Example: $y = \log_2 x$ reflecting the graph of $y = a^x$ when $a = 2$ as shown in Fig.

So that mean

$y = \log_a x$

$a^y = x$



Inverse Equations for a^x and $\log_a x$

1. $a^{\log_a x} = x$ $\quad x > 0$
2. $\log_a (a^x) = x$ $\quad$ all x

Rules for base a logarithms for any numbers $x > 0$ and $y > 0$

1. Product Rule: $\log_a xy = \log_a x + \log_a y$

2. Quotient Rule: $\log_a \frac{x}{y} = \log_a x - \log_a y$

3. Reciprocal Rule: $\log_a \frac{1}{y} = -\log_a y$

4. Power Rule: $\log_a x^y = y \log_a x$

Also

1. $\log_a a = 1$
2. $\log_a 1 = 0$
3. $\log_a x = \dfrac{\ln x}{\ln a}$
4. $\log_a x = \dfrac{\log x}{\log a}$
5. $\log x = \dfrac{\ln x}{\ln 10}$

EXAMPLE 1: Prove that $\log_a x = \dfrac{\ln x}{\ln a}$

Proof: $a^{\log_a x} = x$

tak ln $\Rightarrow \ln a^{\log_a x} = \ln x$ using properties

$$\log_a x \cdot \ln a = \ln x$$

$$\log_a x = \frac{\ln x}{\ln a}$$

EXAMPLE 2: Calculate $\dfrac{1}{\log_{10} 30} + \dfrac{1}{\log_3 30}$

Sol:

$$\frac{1}{\frac{\log 30}{\log 10}} + \frac{1}{\frac{\log 30}{\log 3}} \quad or \quad \frac{1}{\frac{\ln 30}{\ln 10}} + \frac{1}{\frac{\ln 30}{\ln 3}}$$

$$\Rightarrow \frac{\ln 10}{\ln 30} + \frac{\ln 3}{\ln 30} = \frac{\ln 10 + \ln 3}{\ln 30}$$

$$= \frac{\ln(10 \times 3)}{\ln 30} = \frac{\ln 30}{\ln 30} = 1$$

EXAMPLE 3: Evaluate

1. $2^{\log_2(3)}$

Sol: $a^{\log_a x} = x$

$2^{\log_2(3)} = 3$

2. $\log_{10}(10)^{-7}$

Sol:

$\log_{10}(10)^{-7} \Rightarrow \log_a a^x = x$

$\log_{10}(10)^{-7} = -7$

3. $\log_2(1/4) = -2$ (H.W)

4. $\log_a x^n = \dfrac{\ln x^n}{\ln a} = n\dfrac{\ln x}{\ln a} = n\log_a x$

EXAMPLE 4: If $y = \log_3 x$ find the value of x

Sol:

$y = \log_a x \quad \Rightarrow a^y = x$

$3^y = x \quad \Rightarrow x = 3^y$

EXAMPLE 5: calculate $y = \log_4 8$

Sol:

$x = a^y \qquad 4^y = 8$

$2^{2y} = 2^3$

$2y = 3 \Rightarrow y = 3/2$

H.W Ex 6: Calculate

1. $y = \log_2 8$
2. $y = \log_7 49$

EXAMPLE 7: Find the value of x

1. $\log_3 x = 4$

Sol:

$a^y = x \quad \Rightarrow 3^4 = x$

$x = 81$

2. $\log_{64} x = 6$ (H.W)
3. $\log_5(1/125) = x$ (H.W)

H.W Ex 8: Find value of y

1. $y = 5^{\log_5 7} \quad \Rightarrow \quad y = 7$
2. $y = \log_6 36 \quad \Rightarrow \quad y = 2$
3. $y = \log_3(1/9) \quad \Rightarrow \quad y = -2$
4. $y = \log_3 \sqrt{3} \quad \Rightarrow \quad y = 1/2$
5. $y = \log_4(4^{2/3}) \quad \Rightarrow \quad y = 2/3$
6. $y = \log_x(1/\sqrt{x}) \quad \Rightarrow \quad y = -1/2$

Exponential function

$y = a^x \quad \Rightarrow a : \text{constant}$

$y = e^x \quad \Rightarrow \quad e \approx 2.718$

Properties and rule of Exponential function

1. $e^x \cdot e^y = e^{x+y}$
2. $e^x / e^y = e^{x-y}$
3. $(e^x)^n = e^{nx}$

Rule

1. $\ln e = 1$
2. $\ln 1 = 0$
3. $\ln e^u = u \ln e = u$
4. $e^{\ln 1} = e^0 = 1$
5. $e^{\ln u} = u$

EXAMPLE 1: Solve for $e^{\ln y+2x}=e^{\ln(x+1)}$

Sol:

$$e^{\ln y+2x}=e^{\ln(x+1)}$$

$$e^{\ln y}\cdot e^{2x}=(x+1)$$

$$y\cdot e^{2x}=(x+1)$$

$$y=(x+1)/e^{2x}$$

1.2 <u>Invers function</u>

IF $y=\sin^{-1}x \Rightarrow x=\sin y$

OR $x=\sin y \Rightarrow y=\sin^{-1}x$

EXAMPLE 1: Prove that $\sin^{-1}(-x)=-\sin^{-1}x$

Sol:

Let $y=\sin^{-1}(-x)$

$$-x=\sin y$$

$$x=-\sin y$$

$$y=-\sin^{-1}(x)$$

EXAMPLE 2: Prove that $\sec^{-1}x=\cos^{-1}(1/x)$

Sol:

Let $y=\sec^{-1}x$ L.H.S

$$x=\sec y$$

$$x=1/\cos y$$

$$1/x=\cos y$$

$$y=\cos^{-1}(1/x)$$

So $\sec^{-1}x=\cos^{-1}(1/x)$ R.H.S

Hyperbolic Function

Definition of hyperbolic function

1. $\sinh x = \frac{1}{2}(e^x - e^{-x}) = \frac{(e^x - e^{-x})}{2}$
2. $\cosh x = \frac{1}{2}(e^x + e^{-x}) = \frac{(e^x + e^{-x})}{2}$
3. $\tanh x = \frac{\sinh x}{\cosh x} = \frac{(e^x - e^{-x})}{(e^x + e^{-x})}$
4. $\coth x = \frac{\cosh x}{\sinh x} = \frac{(e^x + e^{-x})}{(e^x - e^{-x})}$
5. $\text{sechx} = \frac{1}{\cosh x} = \frac{2}{(e^x + e^{-x})}$
6. $\text{cschx} = \frac{1}{\sinh x} = \frac{2}{(e^x - e^{-x})}$

Derivatives

Rule of Derivatives: Let c and n are constant, u, v and ware differentiable function of x:

1. $\frac{d}{dx}c = 0$
2. $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$
3. $\frac{d}{dx}(\frac{1}{u}) = -\frac{1}{u^2}\frac{du}{dx}$
4. $\frac{d}{dx}cu = c\frac{du}{dx}$
5. $\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$ and $\frac{d}{dx}(u \cdot v \cdot w) = u \cdot v\frac{dw}{dx} + u \cdot w\frac{dv}{dx} + v \cdot w\frac{du}{dx}$

6. $\frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ where $v \neq 0$

EXAMPLE 1: Find $\frac{dy}{dx}$ for the following function.

1. $y = (x^2 + 1)^5$

Sol: $y = (x^2 + 1)^5$

$y' = 5(x^2 + 1)^4(2x)$

$y' = 10x(x^2 + 1)^4$

2. $y = \frac{x^2 - 1}{x^2 + x - 2}$

Sol:

$$y' = \frac{2x(x^2 + x - 2) - (2x + 1)(x^2 - 1)}{(x^2 + x - 2)^2}$$

$$y' = \frac{2x^3 + 2x^2 - 4x - 2x^3 + 2x - x^2 + 1}{(x^2 + x - 2)^2} = \frac{x^2 - 2x + 1}{(x^2 + x - 2)^2}$$

H.W Ex 3: $y = \frac{12}{x} - \frac{4}{x^2} + \frac{3}{x^4}$

H.W Ex 4: $y = (2x^3 - 3x^2 + 6x)^{-5}$

H.W Ex 5: $y = \frac{x^2 - 1}{x^2 + x - 2}$

EXAMPLE 6: $y = \frac{x^2 - 1}{x + 1}$

Sol:

$$y' = \frac{(x+1)(2x) - (x^2 - 1)}{(x+1)^2} = \frac{2x^2 + 2x - x^2 + 1}{(x+1)^2} = \frac{x^2 + 2x + 1}{(x+1)^2}$$

EXAMPLE 7: $y = \sqrt[3]{x^2} \quad \Rightarrow \quad y = x^{2/3}$

$$y' = \frac{2}{3}x^{-1/3}$$

<u>The chain rule</u>

1. Suppose that $h = g \cdot f$ is the composite of the differentiable functions $y = g(t)$ and $x = f(t)$, then h is a differentiable function of x whose derivative at each value of x is

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}}$$

EXAMPLE 1: Find $\frac{dy}{dx}$ if $y = \frac{1}{t^2+1}$, $x = \sqrt{4t+1}$

Sol:

$$y = (t^2+1)^{-1},\ x = \sqrt{4t+1}$$

$$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{\frac{d}{dt}(t^2+1)^{-1}}{\frac{d}{dt}(4t+1)^{1/2}}$$

$$= \frac{-(t^2+1)^{-2}(2t)}{\frac{1}{2}(4t+1)^{-1/2} \cdot 4} = \frac{-2t\ (t^2+1)^{-2}}{2(4t+1)^{-1/2}}$$

$$= \frac{-t\ (t^2+1)^{-2}}{(4t+1)^{-1/2}}$$

2. If y is a differentiable function of t and t is a differentiable function of x ,then y is a differentiable of x:

$y = g(t)$ and $t = f(x)$

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}}$$

EXAMPLE 1: Use the chain rule to express $\frac{dy}{dx}$ in terms of x and y

$$y=\frac{t^2}{t^2+1} \qquad , \qquad t=\sqrt{2x+1}=(2x+1)^{1/2}$$

Sol:

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{(t^2+1)2t-t^2(2t)}{(t^2+1)^2}\cdot\frac{1}{2}(2x+1)^{-1/2}(2)$$

$$=\frac{2t^3+2t-2t^3}{(t^2+1)^2}\cdot(2x+1)^{-1/2}$$

$$=\frac{2t}{(t^2+1)^2}\cdot\frac{1}{\sqrt{2x+1}} \qquad \text{sub } t$$

$$=\frac{2\sqrt{2x+1}}{(2x+1+1)^2}\cdot\frac{1}{\sqrt{2x+1}}=\frac{2}{(2x+2)^2}$$

EXAMPLE 2: Use the chain rule to express $\frac{dy}{dx}$ in terms of x and y

$$y=\left(\frac{t-1}{t+1}\right)^2,\ x=\frac{1}{t^2}-1 \quad \text{at} \quad t=2$$

Sol:

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt} \Rightarrow \qquad y=\left(\frac{t-1}{t+1}\right)^2$$

$$\frac{dy}{dt}=2\left(\frac{t-1}{t+1}\right)\frac{t+1-(t-1)}{(t+1)^2}$$

$$\frac{dy}{dt}=\frac{4(t-1)}{(t+1)^3}$$

$$\left.\frac{dy}{dt}\right|_{t=2}=\frac{4(2-1)}{(2+1)^3}=4/27$$

$$x = \frac{1}{t^2} - 1$$

$$\left.\frac{dx}{dt}\right|_{t=2} = \frac{-2}{t^3} = -1/4$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = 4/27 \div (-1/4) = -16/27$$

Higher derivative

If a function $y = f(x)$ possesses a derivative at every point of some interval. We may form the function $f'(x)$ and take about its derivate if it has one.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx} f'(x)$$

This derivative is called the second derivative of y with respect to x. In some manner we may define third and higher derivatives using similar notations.

EXAMPLE 1: Find all derivatives of the following function.

$$y = 3x^3 - 4x^2 + 7x + 10$$

Sol:

$$y' = 9x^2 - 8x + 7$$

$$y'' = 18x - 8$$

$$y''' = 18$$

$$y'''' = 0$$

EXAMPLE 2:

$$y = \frac{1}{x} + \sqrt{x^3} \quad \Rightarrow y = x^{-1} + x^{3/2}$$

Sol:

$$y' = -\frac{1}{x^2} + \frac{3}{2} x^{1/2}$$

$$y'' = \frac{2}{x^3} + \frac{3}{4}x^{-1/2}$$

$$y''' = -\frac{6}{x^4} - \frac{3}{8}x^{-3/2}$$

$$= -\frac{6}{x^4} - \frac{3}{8\sqrt{x^3}}$$

Implicit derivative

If the formula of f is an algebraic combination of power of x and y.To calculate the derivative of the implicitly defined functions. We simply differentiable both sides of the defining equation with respect to x.

EXAMPLE 1: Find $\frac{dy}{dx}$ for the following function.

1. $x^2 y^2 = x^2 + y^2$

Sol:

$$x^2 2yy' + 2xy^2 = 2x + 2y\,y'$$

$$x^2 2yy' - 2y\,y' = 2x - 2xy^2$$

$$y'(2x^2 y - 2y) = 2x - 2xy^2$$

$$y' = \frac{2x - 2xy^2}{2x^2 y - 2y} = \frac{x - xy^2}{x^2 y - y}$$

2. $\frac{x-y}{x-2y} = 2$

Sol:

$$2x - 4y = x - y$$

$$2 - 4y' = 1 - y'$$

$$2 - 1 = -y' + 4y'$$

$$1 = 3y'$$

$$y' = 1/3$$

3. $xy + 2x - 5y = 2$ at $(3,2)$

Sol:

$$xy' + y + 2 - 5y' = 0$$

$$y'(x-5) = -y - 2$$

$$y' = \frac{-(y+2)}{(x-5)} = \frac{-(2+2)}{(3-5)} = \frac{-4}{-2} = 2$$

Trigonometric function

1. $\sin u$ $\qquad \frac{d}{x}\sin u = \cos u \frac{du}{dx}$
2. $\cos u$ $\qquad \frac{d}{x}\cos u = -\sin u \frac{du}{dx}$
3. $\tan u$ $\qquad \frac{d}{x}\tan u = \sec^2 u \frac{du}{dx}$
4. $\cot u$ $\qquad \frac{d}{x}\cot u = -\csc^2 u \frac{du}{dx}$
5. $\sec u$ $\qquad \frac{d}{x}\sec u = \sec u \tan u \frac{du}{dx}$
6. $\csc u$ $\qquad \frac{d}{x}\csc u = -\csc u \cot u \frac{du}{dx}$

EXAMPLE 1: Prove that $\frac{d}{x}\tan u = \sec^2 u \frac{du}{dx}$

Sol:

$$\frac{d}{x}\tan u = \frac{d}{x}\frac{\sin u}{\cos u}$$

$$= \frac{\cos u \cos u - \sin u(-\sin u)}{\cos^2 u}\frac{du}{dx}$$

$$= \frac{\cos^2 u + \sin^2 u}{\cos^2 u}\frac{du}{dx}$$

$$= \frac{1}{\cos^2 u}\frac{du}{dx} = \sec^2 u \frac{du}{dx}$$

EXAMPLE 2: Prove that $\frac{d}{x}\sec u = \sec u \tan u \frac{du}{dx}$

Sol:

$$\frac{d}{x}\sec u = \frac{d}{dx}\frac{1}{\cos u} = -\frac{1}{\cos^2 u}(-\sin u)\frac{du}{dx}$$

$$= \frac{\sin u}{\cos u}\frac{1}{\cos u}\frac{du}{dx}$$

$$= \sec u \tan u \frac{du}{dx}$$

EXAMPLE 3: Find $\frac{dy}{dx}$ for the following function

1. $y = \tan(3x^2)$

Sol:

$$\frac{dy}{dx} = \sec^2(3x^2)(6x) = 6x\sec^2(3x^2)$$

2. $y = (\csc x + \cot x)^2$

Sol:

$$\frac{dy}{dx} = 2(\csc x + \cot x)(-\csc x \cot x - \csc^2 x)$$

$$= -2\csc x(\csc x + \cot x)^2$$

Hint:

1. $y = \sin^n u$ $\quad y' = n\sin^{n-1} u \cos u \frac{du}{dx}$
2. $y = \cos^n u$ $\quad y' = n\cos^{n-1} u(-\sin u)\frac{du}{dx}$
3. $y = \tan^n u$ $\quad y' = n\tan^{n-1} u \sec^2 u \frac{du}{dx}$
4. $y = \cot^n u$ $\quad y' = n\cot^{n-1} u(-\csc^2 u)\frac{du}{dx}$
5. $y = \sec^n u$ $\quad y' = n\sec^{n-1} u(\sec u \tan u)\frac{du}{dx}$

6. $y = \csc^n u$ $\qquad y' = n\csc^{n-1} u(-\csc u \cot u)\dfrac{du}{dx}$

EXAMPLE 1: Find $\dfrac{dy}{dx}$ for the following function.

1. $y = \tan^2(\cos x)$

$y' = 2\tan(\cos x)\sec^2(\cos x)(-\sin x)$

$y' = -2\sin x \tan(\cos x)\sec^2(\cos x)$

$y = \sec^4 x - \tan^4 x$

$y' = 4\sec^3 x \sec x \tan x - 4\tan^3 x \sec^2 x$

$= 4\sec^4 x \tan x - 4\tan^3 x \sec^2 x$

$= \sec^2 x(4\sec^2 x \tan x - 4\tan^3 x)$

2. $= 2\tan(x/2) - x$

$y' = 2\sec^2(x/2)\cdot(1/2) - 1$

$= \sec^2(x/2) - 1$

$y' = \tan^2(x/2)$

3. $y = \cot^3 x$

$y' = 3\cot^2 x(-\csc^2 x)\cdot 1$

$y' = -3\cot^2 x \csc^2 x$

4. $x + \tan(xy) = 0$

$1 + \sec^2(xy)(xy' + y)$

$\sec^2(xy)xy' + \sec^2(xy)y = -1$

$xy' \sec^2(xy) = -(1 + y\sec^2(xy))$

$$y' = \frac{-(1 + y\sec^2(xy))}{x\sec^2(xy)}$$

5. $y = 2\sin\frac{x}{2} - x\cos\frac{x}{2}$

$y' = 2\cos\frac{x}{2}\cdot\frac{1}{2} - (x(-\sin\frac{x}{2})\cdot\frac{1}{2} + \cos\frac{x}{2})$

$= \cos\frac{x}{2} + \frac{x}{2}\sin\frac{x}{2} - \cos\frac{x}{2} = \frac{x}{2}\sin\frac{x}{2}$

Transcendental function derivative

1- Logarithm function الدالة الوغارتيمية

(1) If $y = \ln x$ $\Rightarrow \frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$

EXAMPLE: $y = \ln x$

$y' = \frac{1}{x}$

(2) $\frac{d}{dx}\log_a u = \frac{d}{dx}(\frac{\ln u}{\ln a}) = \frac{1}{\ln a}\frac{1}{u}\frac{du}{dx}$

EXAMPLE: $y = \log_a x = \frac{\ln x}{\ln a}$ $\Rightarrow y' = \frac{1}{\ln a}\frac{1}{x}$

(3) If $y = \log x = \frac{\ln x}{\ln 10}$ $\Rightarrow y' = \frac{1}{\ln 10}\frac{1}{x}$

$\frac{d}{dx}\log u = \frac{1}{u}\frac{1}{\ln 10}\frac{du}{dx}$

EXAMPLE 1: $y = \ln(\sin x - \sec x)$

$y' = \frac{\cos x - \sec x\tan x}{\sin x - \sec x}$

EXAMPLE 2: Find $\frac{dy}{dx}$ for the following function:

1. $y = \log_{10} e^{x}$

Sol:

$$y = x\log_{10} e \quad \Rightarrow \quad y' = \log_{10} e$$

2. $y = \log_{5}(x+1)^{2}$

Sol:

$$y = 2\log_{5}(x+1) \quad \Rightarrow \quad y = 2\frac{\ln(x+1)}{\ln 5}$$

$$y' = 2\frac{1}{(x+1)\ln 5}$$

3. $y = \log_{2}(3x^{2}+1)^{3}$

Sol:

$$y = 3\log_{2}(3x^{2}+1) \quad \Rightarrow \quad y = 3\frac{\ln(3x^{2}+1)}{\ln 2}$$

$$y' = 3\frac{6x}{(3x^{2}+1)\ \ln 2} = \frac{18x}{(3x^{2}+1)\ \ln 2}$$

4. $y + \ln x + \ln y = 1$ find y'

Sol:

$$y' + \frac{1}{x} + \frac{1}{y}y' = 0$$

$$y'(1+\frac{1}{y}) = -\frac{1}{x}$$

$$y'(\frac{y+1}{y}) = -\frac{1}{x}$$

$$y' = -\frac{y}{x(y+1)}$$

5. $\sin(\ln y) = \ln(x^{2}-3x+1)$

Sol:

$$\cos(\ln y)\cdot\frac{1}{y}y'=\frac{2x-3}{x^2-3x+1}$$

$$\cos(\ln y)\,y'=\frac{y(2x-3)}{x^2-3x+1}$$

$$y'=\frac{y(2x-3)}{\cos(\ln y)(x^2-3x+1)}$$

ملاحظة يمكن استخدام الدالة الوغارتيمية في ايجاد مشتقة الدوال المعقدة او الدوال الاسية

EXAMPLE 7: Find $\frac{dy}{dx}$ for the following function:

1. $y=x^x$

Sol:

$$\ln y=\ln x^x$$

$$\ln y=x\ln x$$

$$\frac{1}{y}y'=x\frac{1}{x}+\ln x \quad \Rightarrow y'=y(1+\ln x)$$

2. $y=x^{\tan x}$

Sol:

$$\ln y=\tan x\ln x$$

$$\frac{1}{y}y'=\tan x\,\frac{1}{x}+\sec^2 x\ln x$$

$$y'=y\ \left(\frac{\tan x}{x}+\sec^2 x\ln x\right)$$

3. $y=\sin x\tan x\cos x\sec x\cot x$

Sol:

$$\ln y=\ln(\sin x\tan x\cos x\sec x\cot x)$$

$$\ln y=\ln\sin x+\ln\tan x+\ln\cos x+\ln\sec x+\ln\cot x$$

$$\frac{1}{y}y' = \frac{\cos x}{\sin x} + \frac{\sec^2 x}{\tan x} + \frac{-\sin x}{\cos x} + \frac{\sec x \tan x}{\sec x} + \frac{-\csc^2 x}{\cot x}$$

$$y' = y(\cot x + \frac{\sec^2 x}{\tan x} - \tan x + \tan x - \frac{\csc^2 x}{\cot x})$$

$$y' = y(\cot x + \sec^2 x \cot x - \csc^2 x \tan x)$$

4. $y = \sqrt[3]{\frac{x \sin x}{(x-1)(x^2+1)}} \qquad \Rightarrow \qquad y = \left(\frac{x \sin x}{(x-1)(x^2+1)}\right)^{1/3}$

Sol:

$$\ln y = \frac{1}{3} \ln \frac{x \sin x}{(x-1)(x^2+1)}$$

$$\ln y = \frac{1}{3}[\ln x + \ln \sin x - (\ln(x-1) + \ln(x^2+1))]$$

$$\frac{1}{y}y' = \frac{1}{3}[\frac{1}{x} + \frac{\cos x}{\sin x} - \frac{1}{(x-1)} - \frac{2x}{(x^2+1)}]$$

$$y' = \frac{y}{3}(\frac{1}{x} + \cot x - \frac{1}{(x-1)} - \frac{2x}{(x^2+1)})$$

2- <u>Exponential function</u> If is u any differentiable function of x then:

1) $\frac{d}{dx} = a^u = a^u \ln a \frac{du}{dx}$

2) $\frac{d}{dx} = e^u = e^u \frac{du}{dx}$

EXAMPLE 7: Find $\frac{dy}{dx}$ for the following function:

1. $y = 2^{3x}$

$$y' = 2^{3x} 3 \ln 2$$

2. $y = (2^x)^2 \quad \Rightarrow \quad y = 2^{2x}$

$$y' = 2^{2x} \ln 2 (2) = 2^{2x+1} \ln 2$$

3. $y = x 2^{x^2}$

$$y' = x 2^{x^2} \ln 2 (2x) + 2^{x^2} = 2^{x^2} (2x^2 \ln 2 + 1)$$

4. $y = e^{\sqrt{1-5x^2}} = e^{(1-5x^2)^{1/2}}$

$$y' = e^{(1-5x^2)^{1/2}} \cdot \frac{1}{2} (1-5x^2)^{-1/2} (10x) = e^{(1-5x^2)^{1/2}} \frac{5x}{\sqrt{1-5x^2}}$$

5. $y = e^{7x}$

$$y' = 7e^{7x}$$

6. $y = e^{\tan x}$

$$y' = e^{\tan x} \sec^2 x$$

7. $y = 3^{\tan x}$

$$y' = \ln 3 \, 3^{\tan x} \sec^2 x$$

8. $y = x 2^{x^2}$

$$y' = x \ln 2 \, 2^{x^2} (2x) + 2^{x^2}$$

9. $e^{(x+y)} = \ln(x^2 + y^2) + \sin x + \tan x$

$$e^{(x+y)} (1 + y') = \frac{2x + 2y y'}{x^2 + y^2} + \cos x + \sec^2 x$$

$$e^{(x+y)} + e^{(x+y)} y' = \frac{2x}{x^2 + y^2} + \frac{2y y'}{x^2 + y^2} + \cos x + \sec^2 x$$

$$e^{(x+y)}y' - \frac{2y\,y'}{x^2+y^2} = \frac{2x}{x^2+y^2} + \cos x + \sec^2 x - e^{(x+y)}$$

$$y'(e^{(x+y)} - \frac{2y}{x^2+y^2}) = \frac{2x}{x^2+y^2} + \cos x + \sec^2 x - e^{(x+y)}$$

$$y' = \frac{\frac{2x}{x^2+y^2} + \cos x + \sec^2 x - e^{(x+y)}}{(e^{(x+y)} - \frac{2y}{x^2+y^2})}$$

10. $y^x = x^y$

$$\ln y^x = \ln x^y$$

$$x \ln y = y \ln x$$

$$x\frac{y'}{y} + \ln y = y\frac{1}{x} + \ln x\, y'$$

$$\frac{x}{y}y' - y'\ln x = \frac{y}{x} - \ln y$$

$$y'(\frac{x}{y} - \ln x) = \frac{y}{x} - \ln y$$

$$y' = \frac{\frac{y}{x} - \ln y}{\frac{x}{y} - \ln x}$$

Inverse function

1. Trigonometric function

(1) $\frac{d}{x}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

(2) $\frac{d}{x}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

(3) $\frac{d}{x}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$

(4) $\frac{d}{x}\cot^{-1}u=-\frac{1}{1+u^2}\frac{du}{dx}$

(5) $\frac{d}{x}\sec^{-1}u=\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad u>1$

(6) $\frac{d}{x}\csc^{-1}u=-\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad u>1$

EXAMPLE 1: Prove that $\frac{d}{x}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}}$

Proof:

Let $y=\sin^{-1}x$

$$x=\sin y$$

$$\frac{d}{dx}x=\frac{d}{x}(\sin y)$$

$$1=\cos y\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{1}{\cos y}$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-\sin^2 y}}$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$$

$$\sin^2 y+\cos^2 y=1$$

$$\cos^2 y=1-\sin^2 y$$

$$\cos y=\sqrt{1-\sin^2 y}$$

$$\cos y=\sqrt{1-x^2}$$

EXAMPLE 2: Prove that $\frac{d}{x}\tan^{-1}x=\frac{1}{1+x^2}$

Proof:

Let $y=\tan^{-1}x$

$$x=\tan y$$

$$\frac{d}{dx}x=\frac{d}{x}(\tan y)$$

$$1 = \sec^2 y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} \qquad\qquad 1 + \tan^2 y = \sec^2 y$$

$$\frac{dy}{dx} = \frac{1}{1 + \tan^2 y}$$

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

EXAMPLE 3: If $y = \tan^{-1}(x^2 - x)$ find $\frac{dy}{dx}$

Sol:

$$y' = \frac{1}{1+u^2} = \frac{2x-1}{1+(x^2-x)^2}$$

EXAMPLE 4: If $y = \sin^{-1} \ln(x)$ find y'

Sol:

$$y' = \frac{1}{\sqrt{1-u^2}} = \frac{1/x}{\sqrt{1-(\ln x)^2}}$$

EXAMPLE 5: $y = e^{\tan^{-1}(3x)}$ find y'

Sol:

$$y' = e^{\tan^{-1}(3x)} \frac{3}{1+(3x)^2}$$

EXAMPLE 6: $y = \ln(e^{\sin^{-1} x} - \tan^{-1} x)$

Sol:

$$y' = \frac{e^{\sin^{-1} x} \frac{1}{\sqrt{1-x^2}} - \frac{1}{1+x^2}}{e^{\sin^{-1} x} - \tan^{-1} x}$$

EXAMPLE 7: If $y = \cot^{-1} 2/x + \tan^{-1} x/2$

Sol:

$$y' = \frac{-(-2/x^2)}{1+(2/x)^2} + \frac{1/2}{1+(x/2)^2}$$

$$y' = \frac{2/x^2}{1+4/x^2} + \frac{1/2}{1+(x/2)^2}$$

EXAMPLE 8: If $y = \sin^{-1}(\frac{x-1}{x+1})$ find y'

Sol:

$$y' = \frac{\frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}}{\sqrt{1-(\frac{x-1}{x+1})^2}} = \frac{\frac{x+1-x+1}{(x+1)^2}}{\sqrt{1-(\frac{x-1}{x+1})^2}}$$

$$y' = \frac{2/(x+1)^2}{\sqrt{1-(\frac{x-1}{x+1})^2}}$$

EXAMPLE 9: If $y = \sec^{-1}(5x)$ find y'

Sol:

$$y' = \frac{5}{5x\sqrt{25x^2-1}}$$

EXAMPLE 10: If $y = x \cdot \ln \sec^{-1} x$ find $\frac{dy}{dx}$

Sol:

$$y' = x\frac{\frac{1}{x\sqrt{x^2-1}}}{\sec^{-1} x} + \ln(\sec^{-1} x)$$

$$y' = \frac{1}{\sqrt{x^2-1}\ \sec^{-1} x} + \ln(\sec^{-1} x)$$

EXAMPLE 11: $y = 3^{\sin^{-1}(2x)} \Rightarrow$ find y'

Sol:

$$y' = \ln 3 \;\; 3^{\sin^{-1}(2x)} \frac{2}{\sqrt{1-4x^2}}$$

<u>Hyperbolic function</u>

If u is any differentiable function of x

1. $\frac{d}{x}\sinh u = \cosh u \frac{du}{dx}$
2. $\frac{d}{x}\cosh u = \sinh u \frac{du}{dx}$
3. $\frac{d}{x}\tanh u = \text{sech}^2 u \frac{du}{dx}$
4. $\frac{d}{x}\coth u = -\text{csch}^2 \frac{du}{dx}$
5. $\frac{d}{x}\text{sech} u = -\text{sech} u \tanh u \frac{du}{dx}$
6. $\frac{d}{x}\text{csch} u = -\text{csch} u \coth u \frac{du}{dx}$

EXAMPLE 1: Find $\frac{dy}{dx}$ for the following function:

1. $y = \coth(\tan x)$

Sol:

$$y' = -\text{csch}^2(\tan x)\sec^2 x$$

2. $y = \sin^{-1}(\tanh x)$

Sol:

$$y' = \frac{\text{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \frac{\text{sech}^2 x}{\sqrt{\text{sech}^2 x}} = \text{sech}\, x$$

3. $y = \ln \tanh x/2$

Sol:

$$y' = \frac{\text{sech}^2 x/2 \cdot (1/2)}{\tanh x/2} = \frac{1}{2}\frac{\frac{1}{\cosh^2 x/2}}{\frac{\sinh x/2}{\cosh x/2}}$$

$$y' = \frac{1}{2\sinh x/2 \cdot \cosh x/2} = \frac{1}{\sinh x} = \text{csch} x$$

4. $y = x\sinh 2x - \frac{1}{2}\cosh 2x$

Sol:

$$y' = x\cosh 2x \cdot 2 + \sinh 2x - \tfrac{1}{2}\sinh 2x \cdot 2$$

$$y' = 2x\cosh 2x$$

5. $y = \text{sech}^3 x$

Sol:

$$y' = 3\text{sech}^2 x(-\text{sech} x \cdot \tanh x)$$

$$y' = -3\text{sech}^3 x \ \tanh x)$$

6. $y = \text{csch}^2 x$

Sol:

$$y' = 2\text{csch} x(-\text{csch} x \coth x)$$

$$y' = -2\text{csch}^2 x \coth x$$

EXAMPLE 2: Prove that $\frac{d}{x}\tanh u = \text{sech}^2 u \frac{du}{dx}$

Sol:

$$\frac{d}{x}\tanh u = \frac{d}{x}(\frac{\sinh u}{\cosh u})$$

$$= \frac{\cosh u \ \cosh u \frac{du}{dx} - \sinh u \sinh u \frac{du}{dx}}{\cosh^2 u}$$

$$= \frac{(\cosh^2 u - \sinh^2 u)\ \frac{du}{dx}}{\cosh^2 u} = \frac{1}{\cosh^2 u}\frac{du}{dx}$$

$$\frac{d}{x}\tanh u = \operatorname{sech}^2 u \frac{du}{dx}$$

EXAMPLE 3: Prove that $\frac{d}{x}\operatorname{sech} u = -\operatorname{sech} u \tanh u \frac{du}{dx}$

Sol:

$$= \frac{d}{dx}\frac{1}{\cosh u} = -\frac{1}{\cosh^2 u}\sinh u \frac{du}{dx}$$

$$= -\operatorname{sech} u \tanh u \frac{du}{dx}$$

<u>The Inverse hyperbolic function</u> If is u any differentiable function of x then:

1. $\frac{d}{x}\sinh^{-1} u = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$

2. $\frac{d}{x}\cosh^{-1} u = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$

3. $\frac{d}{x}\tanh^{-1} u = \frac{1}{1-u^2}\frac{du}{dx} \qquad u < 1$

4. $\frac{d}{x}\coth^{-1} u = \frac{1}{1-u^2}\frac{du}{dx} \qquad u > 1$

5. $\frac{d}{x}\operatorname{sech}^{-1} u = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$

6. $\frac{d}{x}\operatorname{csch}^{-1} u = -\frac{-1}{u\sqrt{1+u^2}}\frac{du}{dx}$

EXAMPLE 1: Find $\frac{dy}{dx}$ for the following function.

1. $y = \cosh^{-1}(\sec x)$

Sol:

$$y' = \frac{\sec x \tan x}{\sqrt{(\sec x)^2 - 1}} = \frac{\sec x \tan x}{\sqrt{\tan^2 x}}$$

$$y' == \frac{\sec x \tan x}{\tan x} = \sec x \qquad \text{where } \tan x > 0$$

2. $y = \tanh^{-1}(\cos x)$

Sol:

$$y' = \frac{-\sin x}{1 - \cos^2 x} = \frac{-\sin x}{\sin^2 x} = -\csc x$$

3. $y = \coth^{-1}(\sec x)$

Sol:

$$y' = \frac{\sec x \tan x}{1 - \sec^2 x} = \frac{\sec x \tan x}{-\tan^2 x} = -\csc x$$

4. $y = \operatorname{sech}^{-1}(\sin 2x)$

Sol:

$$y' = -\frac{2\cos 2x}{\sin 2x\sqrt{1 - \sin^2 2x}} = \frac{-2\cos 2x}{\sin 2x \cos 2x} = -2\csc 2x \qquad \text{where } 2x > 0$$

EXAMPLE 2: Verify the following formulas:

1. $\frac{d}{x}\cosh^{-1} u = \frac{1}{\sqrt{u^2 - 1}}\frac{du}{dx}$

Sol:

Let $y = \cosh^{-1} u$

$u = \cosh y$

$$\frac{du}{dx} = \sinh y \frac{dy}{dx}$$

$$y' = \frac{dy}{dx} = \frac{1}{\sinh y} \frac{du}{dx}$$

$$\cosh^2 y - \sinh^2 y = 1 \qquad \Rightarrow u^2 - \sinh^2 y = 1$$

$$\Rightarrow \sinh y = \sqrt{u^2 - 1}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{u^2 - 1}} \cdot \frac{du}{dx}$$

2. $\frac{d}{x} \tanh^{-1} u = \frac{1}{1 - u^2} \frac{du}{dx}$

Sol:

Let $y = \tanh^{-1} u$

$$u = \tanh y$$

$$\frac{du}{dx} = \operatorname{sech}^2 y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y} \frac{du}{dx}$$

$$\operatorname{sech}^2 y + \tanh^2 y = 1 \qquad \Rightarrow \operatorname{sech}^2 y = 1 - \tanh^2 y = 1 - u^2$$

$$\frac{dy}{dx} = \frac{1}{1 - u^2} \frac{du}{dx}$$

$$\Rightarrow \quad \frac{d}{x} \tanh^{-1} u = \frac{1}{1 - u^2} \frac{du}{dx}$$

مراجعة

((اسئلة اضافية))

EXAMPLE 1: Find $\frac{dy}{dx}$ for the following function.

1. $y = mx + b$

Sol:

$$y' = m$$

2. $y = \frac{1}{x}$

Sol:

$$y' = \frac{-1}{x^2}$$

3. $y = \frac{x}{x-1}$

Sol:

$$y' = \frac{(x-1)(1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

4. $y = \sqrt{x}$

Sol:

$$y = x^{1/2} \Rightarrow y' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

EXAMPLE 2: Find the value of the derivative.

1. $\left.\frac{ds}{dt}\right|_{t=1}$ if $s = 1 - 3t^2$

Sol:

$$\frac{ds}{dt} = 0 - 6t \qquad \left.\frac{ds}{dt}\right|_{t=1} = -6(-1) = 6$$

2. $\left.\frac{dy}{dx}\right|_{x=\sqrt{3}}$ $\quad y = 1 - \frac{1}{x} \quad \Rightarrow y = 1 - x^{-1}$

Sol:

$$\frac{dy}{dx} = 0 - (-1)(x^{-2})$$

$$\frac{dy}{dx} = \frac{1}{x^2} \qquad \Rightarrow \frac{dy}{dx}\bigg|_{x=\sqrt{3}} = \frac{1}{(\sqrt{3})^2} = \frac{1}{3}$$

3. $\frac{dr}{d\theta}\bigg|_{\theta=0}$ if $r = \frac{2}{\sqrt{4-\theta}}$ $\Rightarrow r = 2(4-\theta)^{-1/2}$

Sol:

$$\frac{dr}{d\theta} = -\frac{1}{2}(2)(4-\theta)^{-3/2}(-1)$$

$$\frac{dr}{d\theta}\bigg|_{\theta=0} = \frac{1}{4^{3/2}} = 8$$

EXAMPLE 3: Find the derivative of $y = \frac{t^2-1}{t^2+1}$

Sol:

$$\frac{dy}{dt} = \frac{(t^2+1)(2t)-(t^2-1)(2t)}{(t^2+1)^2}$$

$$\frac{dy}{dt} = \frac{4t}{(t^2+1)^2}$$

EXAMPLE 4: Find an equation for the tangent to the curve $y = x + \frac{2}{x}$ at the point $(1,3)$

Sol:

$$\frac{dy}{dt} = 1 + \frac{-2}{x^2}$$

The slope at $x = 1$

$$y'|_{x=1} = [1 - \frac{2}{x^2}]_{x=1}$$

$$= 1 - 2 = -1$$

$$m = -1$$

The line through $(1,3)$ with slope $m=-1$

$$y-3=(-1)(x-1)$$

$$y=-x+1+3$$

$$y=-x+4$$

EXAMPLE 5: Find higher derivatives $y=x^3-3x^2+2$

Sol:

First $\quad y'=3x^2-6x$

Second $\quad y''=6x-6$

Third $\quad y'''=6$

Fourth $\quad y''''=0$

EXAMPLE 6: Find $\frac{dy}{dx}$ for the following Trigonometric function.

1. $y=\frac{\sin x}{x}$

Sol:

$$y'=\frac{x\cos x-\sin x(1)}{x^2}=\frac{x\cos x-\sin x}{x^2}$$

2. $y=x^2\sin x$

Sol:

$$y'=x^2\cos x+2x\sin x$$

3. $y=\sin x\cos x$

Sol:

$$y'=\sin x(-\sin x)+\cos x\cos x$$

$$y'=\cos^2 x-\sin^2 x$$

4. $y=\frac{\cos x}{1-\sin x}$

Sol:

$$y' = \frac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2}$$

$$y' = \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} = \frac{1-\sin x}{(1-\sin x)^2}$$

$$= \frac{1}{1-\sin x}$$

5. If $y = \sec^2 x$ find y''

Sol:

$$y' = \sec x \tan x$$

$$y'' = \sec x \sec^2 x + \sec x \tan x \tan x$$

$$y'' = \sec^3 x + \sec x \tan^2 x$$

6. Find the slope of the line tangent to the curve $y = \sin^5 x$ at point where $x = \pi/3$

Sol:

$$\frac{dy}{dt} = 5\sin^4 x \cos x$$

$$\left.\frac{dy}{dx}\right|_{x=\pi/3} = 5\left(\frac{\sqrt{3}}{2}\right)^4 \cdot \frac{1}{2} = \frac{45}{32}$$

$\cos \pi/3 = 1/2$

$\sin \pi/3 = \sqrt{3}/2$

7. If $x = 2t+3$ and $y = t^2 - 1$ Find the value of $\frac{dy}{dx}$ at $t = 6$

Sol:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{2} = t$$

$$\frac{dy}{dx} = 6 \qquad t = 6$$

Hint: Note that we are also able to find $\frac{dy}{dx}$ as a function of x

$x = 2t+3$ $\qquad$ $y = t^2 - 1$

$x - 3 = 2t$

$t = \frac{x-3}{2}$

8. Find $\frac{dy}{dx}$ if $y^2 = x$

Sol:

$$2y\,y' = 1$$

$$y' = \frac{1}{2y}$$

9. Find the slope of circle $x^2 + y^2 = 25$ at the point $(3,-4)$

Sol:

$$2x + 2y\,y' = 0$$

$$y' = \frac{-2x}{2y} \quad \Rightarrow \quad y' = -\frac{x}{y}$$

$$y'\big|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}$$

10. Find $\frac{dy}{dx}$ if $y^2 = x^2 + \sin xy$

Sol:

$$2y\,y' = 2x + \cos xy\,(x\,y' + y)$$

$$2y\,y' = 2x + x\,y'\cos xy + y\cos xy$$

$$2y\,y' - x\,y'\cos xy = 2x + y\cos xy$$

$$y'(2y - x\cos xy) = 2x + y\cos xy$$

$$y' = \frac{2x + y\cos xy}{2y - x\cos xy}$$

11. Find $\frac{d}{dx}(\cos x)^{-1/5}$

Sol:

$$-\frac{1}{5}(\cos x)^{-6/5}(-\sin x)$$

$$\frac{1}{5}\sin x(\cos x)^{-6/5}$$

12. Find $\frac{d}{dx}\ln 2x$

Sol:

$$y' = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{2x}\cdot 2 = \frac{1}{x}$$

13.Find $\frac{d}{dx}\ln(x^2+3)$

Sol:

$$y' = \frac{1}{x^2+1}\cdot(2x) = \frac{2x}{x^2+1}$$

14.Find $\frac{d}{dx}\ln x^r$

Sol:

$$y' = \frac{1}{x^r}\cdot rx^{r-1}$$

$$y' = \frac{1}{x^r} r\, x^r\, x^{-1} = \frac{r}{x}$$

15.Find $\frac{dy}{dx}$ if $y = \frac{(x^2+1)(x+3)^{1/2}}{x-1} \qquad x>1$

Sol: we take natural logarithm of the both side

$$\ln y = \ln\frac{(x^2+1)(x+3)^{1/2}}{x-1}$$

$$\ln y = \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1)$$

$$\ln y = \ln(x^2+1) + \tfrac{1}{2}\ln(x+3) - \ln(x-1)$$

$$\frac{y'}{y} = \frac{2x}{(x^2+1)} + \frac{1}{2}\frac{1}{(x+3)} - \frac{1}{(x-1)}$$

$$y' = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right)$$

16. Find $\frac{dy}{dx}$ if 1) $y = 5e^x$ 2) $y = e^{-x}$ 3) $y = e^{\sin x}$

Sol:

1) $y' = 5e^{x}$

2) $y' = -e^{-x}$

3) $y' = e^{\sin x}\cos x$

17.Find $\frac{dy}{dx}$ if 1) $y = x^{\sqrt{2}}$ 2) $y = (2+\sin 3x)^{\pi}$

Sol:

1) $y' = \sqrt{2}\, x^{\sqrt{2}-1}$

2) $y' = \pi(2+\sin 3x)^{\pi-1}(\cos 3x)\cdot 3$

$y' = 3\pi\,(2+\sin 3x)^{\pi-1}(\cos 3x)$

18. Find $\frac{dy}{dx}$ if 1) $y = 3^{x}$ 2) $y = 3^{-x}$ 3) $y = 3^{\sin x}$

Sol:

1) $y' = 3^{x}\ln 3$

2) $y' = -3^{x}\ln 3$

3) $y' = 3^{\sin x}(\ln 3)\cos x$

19. Find $\frac{dy}{dx}$ if $y = x^{x}$ $x > 0$

Sol:

$\ln y = \ln x^{x}$

$\ln y = x\ln x$

$\frac{y'}{y} = (x\frac{1}{x} + \ln x)$

$y' = y\,(1+\ln x)$

$y' = x^{x}\,(1+\ln x)$

Or write x^{x} as a power of e

$y = x^{x} = e^{x\ln x}$

$$\frac{dy}{dx} = \frac{d}{dx} e^{x \ln x}$$

$$\frac{dy}{dx} = e^{x \ln x} (\frac{d}{dx} x \ln x)$$

$$\frac{dy}{dx} = x^x \ (x \frac{1}{x} + \ln x)$$

$$= x^x \ (1 + \ln x)$$

20. Find $\frac{d}{dx} \log_{10}(3x+1)$

Sol:

$$\frac{d}{dx} \log_{10}(3x+1)$$

$$\frac{d}{dx} \frac{\ln (3x+1)}{\ln 10}$$

$$\frac{1}{\ln 10} \frac{3}{3x+1}$$

21.Find $\frac{d}{dt} \tanh \sqrt{1+t^2}$

Sol:

$$= \text{sech}^2 \sqrt{1+t^2} \ \frac{d}{dt} (1+t^2)^{1/2}$$

$$= \text{sech}^2 \sqrt{1+t^2} \ \frac{1}{2} \frac{1}{\sqrt{1+t^2}} \cdot 2t$$

$$= \text{sech}^2 \sqrt{1+t^2} \ \ \frac{t}{\sqrt{1+t^2}}$$

Hint:

1) $\text{sech}^{-1} x = \cosh^{-1}(1/x)$
2) $\text{csch}^{-1} x = \sinh^{-1}(1/x)$
3) $\coth^{-1} x = \tanh^{-1}(1/x)$

22. Find y' or $\frac{dy}{dx}$ $\sin^{-1}(x\,y)=\cos^{-1}(x-y)$

Sol:

$$\frac{xy'+y}{\sqrt{1-(xy)^2}}=\frac{-(1-y')}{\sqrt{1-(x-y)^2}}$$

$$\frac{xy'}{\sqrt{1-(xy)^2}}+\frac{y}{\sqrt{1-(xy)^2}}=\frac{-1}{\sqrt{1-(x-y)^2}}+\frac{y'}{\sqrt{1-(x-y)^2}}$$

$$y'[\frac{x}{\sqrt{1-(xy)^2}}-\frac{1}{\sqrt{1-(x-y)^2}}]=-\frac{1}{\sqrt{1-(x-y)^2}}-\frac{y}{\sqrt{1-(xy)^2}}$$

$$y'=\frac{-\frac{1}{\sqrt{1-(x-y)^2}}-\frac{y}{\sqrt{1-(xy)^2}}}{\frac{x}{\sqrt{1-(xy)^2}}-\frac{1}{\sqrt{1-(x-y)^2}}}$$

$$y'=\frac{\frac{-\sqrt{1-(xy)^2}-y\sqrt{1-(x-y)^2}}{\sqrt{1-(x-y)^2}\,\sqrt{1-(xy)^2}}}{\frac{x\sqrt{1-(x-y)^2}-\sqrt{1-(xy)^2}}{\sqrt{1-(xy)^2}\,\sqrt{1-(x-y)^2}}}$$

$$y'=\frac{-\sqrt{1-(xy)^2}-y\sqrt{1-(x-y)^2}}{x\sqrt{1-(x-y)^2}-\sqrt{1-(xy)^2}}$$

<u>H.W Derivative</u>

1) Find $\frac{dy}{dx}$ for the following function

1. $y=\csc^{-2/3}\sqrt{5x}$ ans: $\frac{5}{3\sqrt{5x}}\cdot\frac{\cot\sqrt{5x}}{\csc^{2/3}\sqrt{5x}}$

2. $y=(x-3)(1-x)$ ans: $4-2x$

3. $y=\dfrac{ax+b}{x}$ ans: $-\dfrac{b}{x^2}$

4. $y=\ln(\cos x)$ ans: $-\tan x$

5. $y=\tan x\sin x$ ans: $\sin x+\tan x\sec x$

6. $y=\dfrac{3x-4}{2x-3}$ ans: $\dfrac{1}{(2x+3)^2}$

7. $y=(\sqrt{x^3}-\dfrac{1}{\sqrt{x}})^2$ ans: $\dfrac{3(x^5-1)}{x^4}$

8. $y=\dfrac{\cos x}{x}$ ans: $\dfrac{-x\cdot\sin x+\cos x}{x^2}$

9. $y=\tan(\sec x)$ ans: $\sec^2(\sec x)\sec x\cdot\tan x$

10. $y=x^2\sin x$ ans: $x^2\cos x+2x\cdot\sin x$

11. $y=\sin^{-1}(5x^2)$ ans: $\dfrac{10x}{\sqrt{1-25x^4}}$

12. $y=\cot^3(\dfrac{x+1}{x-1})$ ans: $\dfrac{6}{(x-1)^2}\cot^2(\dfrac{x+1}{x-1})\csc^2(\dfrac{x+1}{x-1})$

13. $y=\sin(\ln x)+\cos(\ln x)$ ans: $2\cos(\ln x)$

14. $y=\cot^{-1}(\dfrac{1+x}{1-x})$ ans: $-\dfrac{1}{1+x^2}$

15. $y=\tan^{-1}\sqrt{4x^3-2}$ ans: $-\dfrac{6x^2}{(4x^3-1)\sqrt{4x^3-2}}$

16. $y=\sec^{-1}(3x^2+1)^3$ ans: $\dfrac{18x}{(3x^2+1)\sqrt{(3x^2+1)^6-1}}$

17. $y=\sin^{-1}2x\cos^{-1}2x$ ans: $\dfrac{2(\cos^{-1}2x-\sin^{-1}2x)}{\sqrt{1-4x^2}}$

18. $y=\tan^{-1}\ln x$ ans: $\dfrac{1}{x(1+(\ln x)^2)}$

19. $y=(\cos x)^{\sqrt{x}}$ ans: $\dfrac{y}{2\sqrt{x}}(\ln\cos x-2x\tan x)$

20. $y=(\sin x)^{\tan x}$ ans: $y(1+\sec^2 x\ln\sin x)$

21. $y=\sqrt{2x^2+\cosh^2(5x)}$ ans: $\dfrac{2x+5\cosh(5x)\cdot\sinh(5x)}{\sqrt{2x^2+\cosh^2(5x)}}$

22. $y=\sinh(\cos 2x)$ ans: $-2\sin 2x\cosh(\cos 2x)$

23. $y=\csc(1/x)$ ans: $\dfrac{1}{x^2}\cdot\csc(1/x)\cdot\cot(1/x)$

24. $y=x^2\cdot\tanh^2\sqrt{x}$ ans: $x\cdot\tanh\sqrt{x}(\sqrt{x}\,\text{sech}^2\sqrt{x}+2\tanh\sqrt{x})$

25. $y=\ln\dfrac{\sin x\cos x+\tan^3 x}{\sqrt{x}}$ ans: $\dfrac{\cos^2 x-\sin^2 x+3\tan^2 x}{\sin x\cos x+\tan^3 x}-\dfrac{1}{2x}$

26. $y=\log_4\sin x$ ans: $\dfrac{\cot x}{\ln 4}$

27. $y=e^{(x^2-e^{5x})}$ ans: $(2x-5e^{5x})e^{(x^2-e^{5x})}$

28. $y=e^{x^2\tan x}$ ans: $(x^2\sec^2 x+2x\tan x)e^{x^2\tan x}$

29. $y=7^{\csc\sqrt{2x+3}}$ ans: $\dfrac{-7^{\csc\sqrt{2x+3}}\ln 7\csc\sqrt{2x+3}\,\cot\sqrt{2x+3}}{\sqrt{2x+3}}$

30. $y=[\ln(x^2+2)^2]\cos x$ ans: $\dfrac{4x\cos x}{x^2+2}-2\ln(x^2+2)\sin x$

31. $y=\sinh^{-1}(\tan x)$ ans: $\sec x$

32. $y=\sqrt{1+(\ln x)^2}$ ans: $\dfrac{\ln x}{x\sqrt{1+(\ln x)^2}}$

33. $\dfrac{e^x}{\ln x}$ ans: $\dfrac{e^x(\ln x-1)}{x(\ln x)^2}$

34. $x^3 \log_2(3-2x)$ ans: $2x^2 \log_2(3-2x) - \dfrac{2x^3}{(3-2x)\ln 2}$

35. $y = 2\cosh^{-1}\dfrac{x}{2} + \dfrac{x}{2}\sqrt{x^2-4}$ ans: $\dfrac{x^2}{\sqrt{x^2-4}}$

2) Verify the following derivative

1. $\dfrac{d}{dx}[5x + (\sqrt{x} + \dfrac{1}{\sqrt{x}})^2 = 6 - \dfrac{1}{x^2}$

2. $\dfrac{d}{dx}[\sqrt{x} + (ax^2 + bx + c)] = \dfrac{1}{2\sqrt{x}}(5ax^2 + 3bx + c)$

3) Find the derivative of y with respect to x in the following function:

1. $y = \dfrac{u^2}{u^2+1}$ and $u = 3x^2 - 2$ ans: $\dfrac{18x^2y^2}{(3x^3-2)^3}$

2. $y = \sqrt{u} + 2u$ and $u = x^2 - 3$ ans: $\dfrac{x}{\sqrt{x^2-3}} + 4x$

4) Find the second derivative for the following function

1. $y = (x + \dfrac{1}{x})^3$ ans: $6x + \dfrac{6}{x^3} + \dfrac{12}{x^5}$

2. $y = \sqrt{2x} + \dfrac{2\sqrt{2}}{\sqrt{x}}$ at $x = 2$ ans: $\dfrac{1}{4}$

3. $y = x^2 - 2xy + y^2 - 16x = 0$ ans: $\pm x^{-3/2}$

5) Find the third derivative of the function $y = \sqrt{x^3}$ ans: $-\dfrac{3}{8y}$

6) Find $\dfrac{dy}{dx}$ for the following implicit function:

1. $\sqrt{xy} + 1 = y$ ans: $\dfrac{y}{2\sqrt{xy} - x}$

2. $\sinh y = \tan^2 x$ ans: $\dfrac{2\tan x \cdot \sec^2 x}{\cosh y}$

3. $x^3 + 4x\sqrt{y} - \dfrac{5y^2}{x} = 3$ ans: $\dfrac{3x^2 + 5y^2x^{-2} + 4\sqrt{y}}{10x^{-1}y - (2x/\sqrt{y})}$

4. $3xy = (x^3 + y^3)^{3/2}$ ans: $\dfrac{3x^2\sqrt{x^3 + y^3} - 2y}{2x - 3y^2\sqrt{x^3 + y^3}}$

5. $\sin^{-1}(xy) = \cos^{-1}(x - y)$ ans: $\dfrac{y\sqrt{1-(x-y)^2} + \sqrt{1-(xy)^2}}{\sqrt{1-(xy)^2} - x\sqrt{1-(x-y)^2}}$

6. $y^2\sin(xy) = \tan x$ ans: $\dfrac{\sec^2 x - y^3\cos(xy)}{2y\sin(xy) + xy^2\cos(xy)}$

7. $x^3 + x\tan^{-1} y = y$ ans: $\dfrac{(1+y^2)(3x^2 + \tan^{-1} y)}{1 + y^2 - x}$

7) Prove the following function

1. $\dfrac{d}{x}\cot u = -\csc^2 u\dfrac{du}{dx}$

2. $\dfrac{d}{x}\csc u = -\csc u\cot u\dfrac{du}{dx}$

3. $\dfrac{d}{x}\cosh^{-1} u = -\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$

4. $\dfrac{d}{x}\mathrm{sec}h^{-1} u = \dfrac{1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}$

5. $\dfrac{d}{x}\sinh u = \cosh u\dfrac{du}{dx}$

6. $\frac{d}{x}\operatorname{csch}u = -\operatorname{csch}u \coth u \frac{du}{dx}$

7. $\frac{d}{x}\sinh^{-1}u = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}$

8. $\frac{d}{x}\operatorname{sech}^{-1}u = \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx}$