Matrix Algebra

2.1 INTRODUCTION

This chapter introduces the basic elements of matrix algebra used in the remainder of this book. It is essentially a review of the requisite matrix tools and is not intended to be a complete development. However, it is sufficiently self-contained so that those with no previous exposure to the subject should need no other reference. Anyone unfamiliar with matrix algebra should plan to work most of the problems entailing numerical illustrations. It would also be helpful to explore some of the problems involving general matrix manipulation.

With the exception of a few derivations that seemed instructive, most of the results are given without proof. Some additional proofs are requested in the problems. For the remaining proofs, see any general text on matrix theory or one of the specialized matrix texts oriented to statistics, such as Graybill (1969), Searle (1982), or Harville (1997).

2.2 NOTATION AND BASIC DEFINITIONS

2.2.1 Matrices, Vectors, and Scalars

A *matrix* is a rectangular or square array of numbers or variables arranged in rows and columns. We use uppercase boldface letters to represent matrices. All entries in matrices will be real numbers or variables representing real numbers. The elements of a matrix are displayed in brackets. For example, the ACT score and GPA for three students can be conveniently listed in the following matrix:

$$\mathbf{A} = \begin{pmatrix} 23 & 3.54 \\ 29 & 3.81 \\ 18 & 2.75 \end{pmatrix}. \tag{2.1}$$

The elements of **A** can also be variables, representing possible values of ACT and GPA for three students:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}. \tag{2.2}$$

In this double-subscript notation for the elements of a matrix, the first subscript indicates the row; the second identifies the column. The matrix **A** in (2.2) can also be expressed as

$$\mathbf{A} = (a_{ij}), \tag{2.3}$$

where a_{ij} is a general element.

With three rows and two columns, the matrix **A** in (2.1) or (2.2) is said to be 3×2. In general, if a matrix **A** has n rows and p columns, it is said to be $n \times p$. Alternatively, we say the *size* of **A** is $n \times p$.

A *vector* is a matrix with a single column or row. The following could be the test scores of a student in a course in multivariate analysis:

$$\mathbf{x} = \begin{pmatrix} 98 \\ 86 \\ 93 \\ 97 \end{pmatrix}. \tag{2.4}$$

Variable elements in a vector can be identified by a single subscript:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}. \tag{2.5}$$

We use lowercase boldface letters for column vectors. Row vectors are expressed as

$$\mathbf{x}' = (x_1, x_2, x_3, x_4) \quad \text{or as} \quad \mathbf{x}' = (x_1 \quad x_2 \quad x_3 \quad x_4),$$

where $\mathbf{x}'$ indicates the *transpose* of **x**. The transpose operation is defined in Section 2.2.3.

Geometrically, a vector with p elements identifies a point in a p-dimensional space. The elements in the vector are the coordinates of the point. In (2.35) in Section 2.3.3, we define the distance from the origin to the point. In Section 3.12, we define the distance between two vectors. In some cases, we will be interested in a directed line segment or arrow from the origin to the point.

A single real number is called a *scalar*, to distinguish it from a vector or matrix. Thus 2, −4, and 125 are scalars. A variable representing a scalar is usually denoted by a lowercase nonbolded letter, such as $a = 5$. A product involving vectors and matrices may reduce to a matrix of size 1×1, which then becomes a scalar.

2.2.2 Equality of Vectors and Matrices

Two matrices are equal if they are the same size and the elements in corresponding positions are equal. Thus if $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$, then $\mathbf{A} = \mathbf{B}$ if $a_{ij} = b_{ij}$ for all i and j. For example, let

$$\mathbf{A} = \begin{pmatrix} 3 & -2 & 4 \\ 1 & 3 & 7 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3 & 1 \\ -2 & 3 \\ 4 & 7 \end{pmatrix},$$
$$\mathbf{C} = \begin{pmatrix} 3 & -2 & 4 \\ 1 & 3 & 7 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 3 & -2 & 4 \\ 1 & 3 & 6 \end{pmatrix}.$$

Then $\mathbf{A} = \mathbf{C}$. But even though $\mathbf{A}$ and $\mathbf{B}$ have the same elements, $\mathbf{A} \neq \mathbf{B}$ because the two matrices are not the same size. Likewise, $\mathbf{A} \neq \mathbf{D}$ because $a_{23} \neq d_{23}$. Thus two matrices of the same size are unequal if they differ in a single position.

2.2.3 Transpose and Symmetric Matrices

The *transpose* of a matrix $\mathbf{A}$, denoted by $\mathbf{A}'$, is obtained from $\mathbf{A}$ by interchanging rows and columns. Thus the columns of $\mathbf{A}'$ are the rows of $\mathbf{A}$, and the rows of $\mathbf{A}'$ are the columns of $\mathbf{A}$. The following examples illustrate the transpose of a matrix or vector:

$$\mathbf{A} = \begin{pmatrix} -5 & 2 & 4 \\ 3 & 6 & -2 \end{pmatrix}, \quad \mathbf{A}' = \begin{pmatrix} -5 & 3 \\ 2 & 6 \\ 4 & -2 \end{pmatrix},$$
$$\mathbf{B} = \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix}, \quad \mathbf{B}' = \begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix},$$
$$\mathbf{a} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \quad \mathbf{a}' = (2, \quad -3, \quad 1).$$

The transpose operation does not change a scalar, since it has only one row and one column.

If the transpose operator is applied twice to any matrix, the result is the original matrix:

$$(\mathbf{A}')' = \mathbf{A}. \tag{2.6}$$

If the transpose of a matrix is the same as the original matrix, the matrix is said to be *symmetric*; that is, $\mathbf{A}$ is symmetric if $\mathbf{A} = \mathbf{A}'$. For example,

$$\mathbf{A} = \begin{pmatrix} 3 & -2 & 4 \\ -2 & 10 & -7 \\ 4 & -7 & 9 \end{pmatrix}, \quad \mathbf{A}' = \begin{pmatrix} 3 & -2 & 4 \\ -2 & 10 & -7 \\ 4 & -7 & 9 \end{pmatrix}.$$

Clearly, all symmetric matrices are square.

2.2.4 Special Matrices

The *diagonal* of a $p \times p$ square matrix $\mathbf{A}$ consists of the elements $a_{11}, a_{22}, \ldots, a_{pp}$. For example, in the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & -2 & 4 \\ 7 & 9 & 3 \\ -6 & 8 & 1 \end{pmatrix},$$

the elements 5, 9, and 1 lie on the diagonal. If a matrix contains zeros in all off-diagonal positions, it is said to be a *diagonal matrix*. An example of a diagonal matrix is

$$\mathbf{D} = \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}.$$

This matrix can also be denoted as

$$\mathbf{D} = \text{diag}(10, -3, 0, 7). \tag{2.7}$$

A diagonal matrix can be formed from any square matrix by replacing off-diagonal elements by 0's. This is denoted by diag($\mathbf{A}$). Thus for the preceding matrix $\mathbf{A}$, we have

$$\text{diag}(\mathbf{A}) = \text{diag}\begin{pmatrix} 5 & -2 & 4 \\ 7 & 9 & 3 \\ -6 & 8 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.8}$$

A diagonal matrix with a 1 in each diagonal position is called an *identity* matrix and is denoted by $\mathbf{I}$. For example, a 3×3 identity matrix is given by

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.9}$$

An *upper triangular matrix* is a square matrix with zeros below the diagonal, such as

$$\mathbf{T} = \begin{pmatrix} 8 & 3 & 4 & 7 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 6 \end{pmatrix}. \tag{2.10}$$

A *lower triangular matrix* is defined similarly.

A vector of 1's is denoted by $\mathbf{j}$:

$$\mathbf{j} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{2.11}$$

A square matrix of 1's is denoted by $\mathbf{J}$. For example, a 3×3 matrix $\mathbf{J}$ is given by

$$\mathbf{J} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{2.12}$$

Finally, we denote a vector of zeros by $\mathbf{0}$ and a matrix of zeros by $\mathbf{O}$. For example,

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{O} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.13}$$

2.3 OPERATIONS

2.3.1 Summation and Product Notation

For completeness, we review the standard mathematical notation for sums and products. The sum of a sequence of numbers $a_1, a_2, \ldots, a_n$ is indicated by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

If the n numbers are all the same, then $\sum_{i=1}^{n} a = a + a + \cdots + a = na$. The sum of all the numbers in an array with double subscripts, such as

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23}, \end{matrix}$$

is indicated by

$$\sum_{i=1}^{2} \sum_{j=1}^{3} a_{ij} = a_{11} + a_{12} + a_{13} + a_{21} + a_{22} + a_{23}.$$

This is sometimes abbreviated to

$$\sum_{i=1}^{2} \sum_{j=1}^{3} a_{ij} = \sum_{ij} a_{ij}.$$

The product of a sequence of numbers $a_1, a_2, \ldots, a_n$ is indicated by

$$\prod_{i=1}^{n} a_i = (a_1)(a_2)\cdots(a_n).$$

If the n numbers are all equal, the product becomes $\prod_{i=1}^{n} a = (a)(a)\cdots(a) = a^n$.

2.3.2 Addition of Matrices and Vectors

If two matrices (or two vectors) are the same size, their *sum* is found by adding corresponding elements; that is, if $\mathbf{A}$ is $n \times p$ and $\mathbf{B}$ is $n \times p$, then $\mathbf{C} = \mathbf{A} + \mathbf{B}$ is also $n \times p$ and is found as $(c_{ij}) = (a_{ij} + b_{ij})$. For example,

$$\begin{pmatrix} -2 & 5 \\ 3 & 1 \\ 7 & -6 \end{pmatrix} + \begin{pmatrix} 3 & -2 \\ 4 & 5 \\ 10 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 7 & 6 \\ 17 & -9 \end{pmatrix},$$

$$\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 10 \end{pmatrix}.$$

Similarly, the *difference* between two matrices or two vectors of the same size is found by subtracting corresponding elements. Thus $\mathbf{C} = \mathbf{A} - \mathbf{B}$ is found as $(c_{ij}) = (a_{ij} - b_{ij})$. For example,

$$(3 \quad 9 \quad -4) - (5 \quad -4 \quad 2) = (-2 \quad 13 \quad -6).$$

If two matrices are identical, their difference is a zero matrix; that is, $\mathbf{A} = \mathbf{B}$ implies $\mathbf{A} - \mathbf{B} = \mathbf{O}$. For example,

$$\begin{pmatrix} 3 & -2 & 4 \\ 6 & 7 & 5 \end{pmatrix} - \begin{pmatrix} 3 & -2 & 4 \\ 6 & 7 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Matrix addition is commutative:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}. \tag{2.14}$$

The transpose of the sum (difference) of two matrices is the sum (difference) of the transposes:

$$(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}', \tag{2.15}$$

$$(\mathbf{A} - \mathbf{B})' = \mathbf{A}' - \mathbf{B}', \tag{2.16}$$

$$(\mathbf{x} + \mathbf{y})' = \mathbf{x}' + \mathbf{y}', \tag{2.17}$$

$$(\mathbf{x} - \mathbf{y})' = \mathbf{x}' - \mathbf{y}'. \tag{2.18}$$

2.3.3 Multiplication of Matrices and Vectors

In order for the product **AB** to be defined, the number of columns in **A** must be the same as the number of rows in **B**, in which case **A** and **B** are said to be *conformable*. Then the (ij)th element of $\mathbf{C} = \mathbf{AB}$ is

$$c_{ij} = \sum_k a_{ik} b_{kj}. \tag{2.19}$$

Thus c_{ij} is the sum of products of the ith row of **A** and the jth column of **B**. We therefore multiply each row of **A** by each column of **B**, and the size of **AB** consists of the number of rows of **A** and the number of columns of **B**. Thus, if **A** is $n \times m$ and **B** is $m \times p$, then $\mathbf{C} = \mathbf{AB}$ is $n \times p$. For example, if

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 6 & 5 \\ 7 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 8 \end{pmatrix},$$

then

$$\mathbf{C} = \mathbf{AB} = \begin{pmatrix} 2\cdot 1 + 1\cdot 2 + 3\cdot 3 & 2\cdot 4 + 1\cdot 6 + 3\cdot 8 \\ 4\cdot 1 + 6\cdot 2 + 5\cdot 3 & 4\cdot 4 + 6\cdot 6 + 5\cdot 8 \\ 7\cdot 1 + 2\cdot 2 + 3\cdot 3 & 7\cdot 4 + 2\cdot 6 + 3\cdot 8 \\ 1\cdot 1 + 3\cdot 2 + 2\cdot 3 & 1\cdot 4 + 3\cdot 6 + 2\cdot 8 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 38 \\ 31 & 92 \\ 20 & 64 \\ 13 & 38 \end{pmatrix}.$$

Note that **A** is 4×3, **B** is 3×2, and **AB** is 4×2. In this case, **AB** is of a different size than either **A** or **B**.

If **A** and **B** are both $n \times n$, then **AB** is also $n \times n$. Clearly, $\mathbf{A}^2$ is defined only if **A** is a square matrix.

In some cases **AB** is defined, but **BA** is not defined. In the preceding example, **BA** cannot be found because **B** is 3×2 and **A** is 4×3 and a row of **B** cannot be multiplied by a column of **A**. Sometimes **AB** and **BA** are both defined but are different in size. For example, if **A** is 2×4 and **B** is 4×2, then **AB** is 2×2 and **BA** is 4×4. If **A** and **B** are square and the same size, then **AB** and **BA** are both defined. However,

$$\mathbf{AB} \neq \mathbf{BA}, \tag{2.20}$$

except for a few special cases. For example, let

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix}.$$

Then

$$\mathbf{AB} = \begin{pmatrix} 10 & 13 \\ 14 & 16 \end{pmatrix}, \qquad \mathbf{BA} = \begin{pmatrix} -3 & -5 \\ 13 & 29 \end{pmatrix}.$$

Thus we must be careful to specify the order of multiplication. If we wish to multiply both sides of a matrix equation by a matrix, we must multiply *on the left* or *on the right* and be consistent on both sides of the equation.

Multiplication is distributive over addition or subtraction:

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}, \tag{2.21}$$

$$\mathbf{A}(\mathbf{B} - \mathbf{C}) = \mathbf{AB} - \mathbf{AC}, \tag{2.22}$$

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}, \tag{2.23}$$

$$(\mathbf{A} - \mathbf{B})\mathbf{C} = \mathbf{AC} - \mathbf{BC}. \tag{2.24}$$

Note that, in general, because of (2.20),

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) \neq \mathbf{BA} + \mathbf{CA}. \tag{2.25}$$

Using the distributive law, we can expand products such as $(\mathbf{A} - \mathbf{B})(\mathbf{C} - \mathbf{D})$ to obtain

$$\begin{aligned} (\mathbf{A} - \mathbf{B})(\mathbf{C} - \mathbf{D}) &= (\mathbf{A} - \mathbf{B})\mathbf{C} - (\mathbf{A} - \mathbf{B})\mathbf{D} && \text{[by (2.22)]} \\ &= \mathbf{AC} - \mathbf{BC} - \mathbf{AD} + \mathbf{BD} && \text{[by (2.24)]}. \end{aligned} \tag{2.26}$$

The transpose of a product is the product of the transposes in reverse order:

$$(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'. \tag{2.27}$$

Note that (2.27) holds as long as $\mathbf{A}$ and $\mathbf{B}$ are conformable. They need not be square.

Multiplication involving vectors follows the same rules as for matrices. Suppose $\mathbf{A}$ is $n \times p$, $\mathbf{a}$ is $p \times 1$, $\mathbf{b}$ is $p \times 1$, and $\mathbf{c}$ is $n \times 1$. Then some possible products are $\mathbf{Ab}$, $\mathbf{c}'\mathbf{A}$, $\mathbf{a}'\mathbf{b}$, $\mathbf{b}'\mathbf{a}$, and $\mathbf{ab}'$. For example, let

$$\mathbf{A} = \begin{pmatrix} 3 & -2 & 4 \\ 1 & 3 & 5 \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}.$$

Then

$$\mathbf{Ab} = \begin{pmatrix} 3 & -2 & 4 \\ 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 16 \\ 31 \end{pmatrix},$$

$$\mathbf{c}'\mathbf{A} = (2 \quad -5)\begin{pmatrix} 3 & -2 & 4 \\ 1 & 3 & 5 \end{pmatrix} = (1 \quad -19 \quad -17),$$

$$\mathbf{c}'\mathbf{A}\mathbf{b} = (2 \quad -5)\begin{pmatrix} 3 & -2 & 4 \\ 1 & 3 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = (2 \quad -5)\begin{pmatrix} 16 \\ 31 \end{pmatrix} = -123,$$

$$\mathbf{a}'\mathbf{b} = (1 \quad -2 \quad 3)\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = 8,$$

$$\mathbf{b}'\mathbf{a} = (2 \quad 3 \quad 4)\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = 8,$$

$$\mathbf{a}\mathbf{b}' = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}(2 \quad 3 \quad 4) = \begin{pmatrix} 2 & 3 & 4 \\ -4 & -6 & -8 \\ 6 & 9 & 12 \end{pmatrix},$$

$$\mathbf{a}\mathbf{c}' = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}(2 \quad -5) = \begin{pmatrix} 2 & -5 \\ -4 & 10 \\ 6 & -15 \end{pmatrix}.$$

Note that $\mathbf{Ab}$ is a column vector, $\mathbf{c}'\mathbf{A}$ is a row vector, $\mathbf{c}'\mathbf{Ab}$ is a scalar, and $\mathbf{a}'\mathbf{b} = \mathbf{b}'\mathbf{a}$. The triple product $\mathbf{c}'\mathbf{Ab}$ was obtained as $\mathbf{c}'(\mathbf{Ab})$. The same result would be obtained if we multiplied in the order $(\mathbf{c}'\mathbf{A})\mathbf{b}$:

$$(\mathbf{c}'\mathbf{A})\mathbf{b} = (1 \quad -19 \quad -17)\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = -123.$$

This is true in general for a triple product:

$$\mathbf{ABC} = \mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}. \tag{2.28}$$

Thus multiplication of three matrices can be defined in terms of the product of two matrices, since (fortunately) it does not matter which two are multiplied first. Note that $\mathbf{A}$ and $\mathbf{B}$ must be conformable for multiplication, and $\mathbf{B}$ and $\mathbf{C}$ must be conformable. For example, if $\mathbf{A}$ is $n \times p$, $\mathbf{B}$ is $p \times q$, and $\mathbf{C}$ is $q \times m$, then both multiplications are possible and the product $\mathbf{ABC}$ is $n \times m$.

We can sometimes factor a sum of triple products on both the right and left sides. For example,

$$\mathbf{ABC} + \mathbf{ADC} = \mathbf{A}(\mathbf{B} + \mathbf{D})\mathbf{C}. \tag{2.29}$$

As another illustration, let $\mathbf{X}$ be $n \times p$ and $\mathbf{A}$ be $n \times n$. Then

$$\mathbf{X}'\mathbf{X} - \mathbf{X}'\mathbf{AX} = \mathbf{X}'(\mathbf{X} - \mathbf{AX}) = \mathbf{X}'(\mathbf{I} - \mathbf{A})\mathbf{X}. \tag{2.30}$$

2.8 DETERMINANTS

The *determinant* of an $n \times n$ matrix $\mathbf{A}$ is defined as the sum of all $n!$ possible products of n elements such that

1. each product contains one element from every row and every column, and
2. the factors in each product are written so that the column subscripts appear in order of magnitude and each product is then preceded by a plus or minus sign according to whether the number of inversions in the row subscripts is even or odd.

An *inversion* occurs whenever a larger number precedes a smaller one. The symbol $n!$ is defined as

$$n! = n(n-1)(n-2)\cdots 2 \cdot 1. \tag{2.80}$$

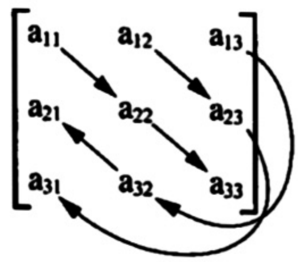
The determinant of $\mathbf{A}$ is a scalar denoted by $|\mathbf{A}|$ or by $\det(\mathbf{A})$. The preceding definition is not useful in evaluating determinants, except in the case of 2×2 or 3×3 matrices. For larger matrices, other methods are available for manual computation, but determinants are typically evaluated by computer. For a 2×2 matrix, the determinant is found by

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}. \tag{2.81}$$

For a 3×3 matrix, the determinant is given by

$$|\mathbf{A}| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{12}a_{21}. \tag{2.82}$$

This can be found by the following scheme. The three positive terms are obtained by



and the three negative terms, by