The integral:-

There are two types of integral:

① definite : which represent the area under curve on an interval $[a, b]$.

② undefinite: which represent the anti-derivative.

Basic integrals:-

① $\int c_1\,dx = c_1 x + c_2$; where c, c_1, c_2 are constants

② $\int c_1 u(x) + c_2 v(x)\, dx = c_1 \int u(x)\, dx + c_2 \int v(x)\, dx$.

③ $\int u^n\, du = \frac{u^{n+1}}{n+1} + c$. , $n \neq -1$

④ $\int e^u\, du = e^u + c$. Ln. , exp. دال

⑤ $\int \frac{du}{u} = \ln|u| + c$.

⑥ $\int \sin u\, du = -\cos u + c$.

⑦ $\int \cos u\, du = \sin u + c$. الدوال المثلثية

⑧ $\int \sec^2 u\, du = \tan u + c$.

⑨ $\int \csc^2 u\, du = -\cot u + c$.

⑩ $\int \sec u \cdot \tan u\, du = \sec u + c$.

⑪ $\int \csc u \cdot \cot u\, du = -\csc u + c$.

⑫ $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + c$.

⑬ $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + c$. الدوال المثلثية العكسية

⑭ $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a} + c$.

Example: Evaluate the integral $\int 2x^3 - \frac{1}{3x^2} + \sqrt[3]{x} + \frac{1}{\sqrt[3]{x^2}}\, dx$

Sol.

$$\int 2x^3 - \frac{1}{3x^2} + \sqrt[3]{x} + \frac{1}{\sqrt[3]{x^2}}\, dx = 2\int x^3 dx - \frac{1}{3}\int x^{-2} dx + \int x^{\frac{1}{3}} dx + \int x^{-\frac{2}{3}} dx$$

$$= \cancel{2} \cdot \frac{x^4}{\cancel{4}} - \frac{1}{3}\,\frac{x^{-1}}{-1} + \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + \frac{x^{\frac{1}{3}}}{\frac{1}{3}} + C$$

$$= \frac{x^4}{2} + \frac{1}{3x} + \frac{3}{4}\sqrt[3]{x^4} + 3\sqrt[3]{x} + C$$

Example: Evaluate the following integrals:

① $\int (x^3+2)^2 \cdot 3x^2\, dx.$

Sol.

$$\int (x^3+2)^2 \cdot 3x^2\, dx = \frac{(x^3+2)^3}{3} + C$$

ملاحظة: اذا توفرت مشتقة داخل القوس يتم اجراء تكامل خارج القوس بشكل جبري.

طريقة ②

Let $u = x^3 + 2 \Rightarrow du = 3x^2\, dx$

$$\therefore \int \underbrace{(x^3+2)^2}_{u^2}\ \underbrace{3x^2\, dx}_{du} = \int u^2\, du = \frac{u^3}{3} + C$$

$$= \frac{(x^3+2)^3}{3} + C$$

② $\int \sqrt{(x^3+2)} \cdot x^2\, dx = \frac{1}{\boxed{3}} \int (x^3+2)^{\frac{1}{2}}\, \boxed{3}\, x^2\, dx$

اذا توفرت مشتقة داخل القوس يتم اجراء التكامل بشكل جبري

$$= \frac{1}{3}\,\frac{(x^3+2)^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{1}{3} \cdot \frac{2}{3}\sqrt{(x^3+2)^3} + C$$

$$= \frac{2}{9}\sqrt{(x^3+2)^3} + C.$$

③ $\int \frac{8x^2}{(x^3+2)^3}\,dx = \int (x^3+2)^{-3} \cdot 8x^2\,dx$

$= \frac{8}{3} \int (x^3+2)^{-3}\, 3x^2\,dx$

$= \frac{8}{3} \frac{(x^3+2)^{-2}}{-2} + c$

$= -\frac{4}{3} \frac{1}{(x^3+2)^2} + c$

$= -\frac{4}{3(x^3+2)^2} + c.$

④ $\int \frac{x^2\,dx}{\sqrt{x^3+2}} = \frac{1}{3} \int (x^3+2)^{-\frac{1}{2}}\, 3x^2\,dx$

$= \frac{1}{3} \cdot \frac{(x^3+2)^{\frac{1}{2}}}{\frac{1}{2}} + C.$

$= \frac{2}{3} \sqrt{x^3+2} + C.$

Q/ Evaluate the integral

$$\int \frac{2(x^3-x)}{\sqrt{x^4-2x^2+1}}\,dx$$

Sol. $\sqrt{x^4-2x^2+1}$.

Example: Evaluate the integrals:

① $\int \frac{dx}{2x-3} = \frac{1}{2}\int \frac{2}{2x-3}\,dx = \frac{1}{2}\cdot \ln|2x-3| + c.$

② $\int \frac{x\,dx}{x^2-1} = \frac{1}{2}\int \frac{2x\,dx}{x^2-1} = \frac{1}{2}\cdot \ln|x^2-1| + c.$

③ $\int \frac{x+2}{x+1}\,dx = \int \frac{(x+1)+1}{x+1}\,dx$

$= \int \left(1 + \frac{1}{x+1}\right) dx$

$= \int dx + \int \frac{1}{x+1}\,dx$

$= x + \ln|x+1| + c$

Example:

① $\int e^{3x}\,dx = \frac{1}{3}\int e^{3x}\, 3\,dx = \frac{e^{3x}}{3} + c.$

② $\int \frac{e^{1/x}\,dx}{x^2} = -\int e^{1/x}\cdot\left(-\frac{1}{x^2}\right) dx$

$= -e^{1/x}\,dx$

③ $\int (e^x+1)^3 e^x\,dx = \frac{(e^x+1)^4}{4} + c.$

④ $\int \frac{dx}{e^x+1} = \int \frac{dx}{e^x(1+e^{-x})} = \int \frac{e^{-x}}{1+e^{-x}}\,dx$

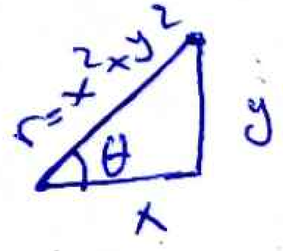
$= -\int \frac{-e^{-x}}{1+e^{-x}}\,dx = \boxed{-\ln(1+e^{-x}) + c}$

$= \boxed{\ln\left(\frac{1}{1+e^{-x}}\right) + c} = \boxed{\ln \frac{e^x}{e^x+1} + c}$

$= \boxed{x - \ln(e^x+1) + c}$

Review of Trigonometrics:.

$r = \sqrt{x^2+y^2}$ is the distance



$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

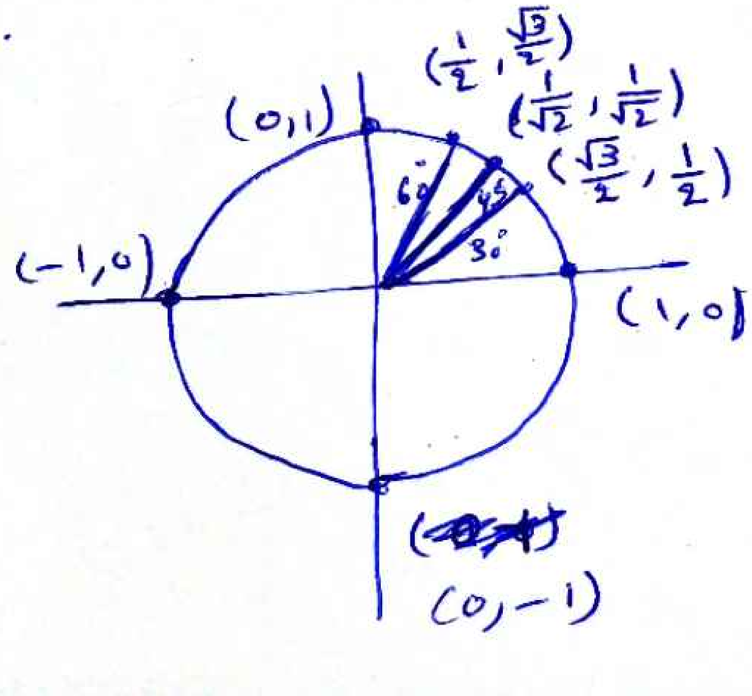
$$\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}.$$

The special angles. $\{30, 45, 60\}$
and $\{0, 90, 180, 270\}$.

Consider the unit circle.



Trigonometric Identity:

1- $\sin(a+b) = \sin a \cdot \cos b + \sin b \cos a$.

2- $\cos(a+b) = \cos a \cdot \cos b - \sin a \sin b$.

3- $\sin 2a = 2\sin a \cos a$.

4- $\cos 2a = \cos^2 a - \sin^2 a$.

5- $\sin^2\theta + \cos^2\theta = 1$

6- $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta = 1 + \tan^2\theta$

7- $\csc^2\theta - \cot^2\theta = 1$ or $\csc^2\theta = 1 + \cot^2\theta$

8- $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

9- $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$.

Radian to degree.

$$\boxed{\begin{aligned} \text{degree} &= \text{rad.} \frac{180}{\pi} \\ \text{rad.} &= \text{degree.} \frac{\pi}{180} \end{aligned}}$$

Example:

① $\int \sin \frac{1}{2} x \, dx = 2 \cdot \int \sin \frac{1}{2} x \left(\frac{1}{2}\right) dx$

$= -2 \cdot \cos \frac{1}{2} x + c$

② $\int \cos 3x \, dx = \frac{1}{3} \int \cos 3x \, (3) \, dx$

$= \frac{1}{3} \sin 3x + c$

③ $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx$

$= \boxed{-\ln|\cos x| + c}$

$= \ln \frac{1}{|\cos x|} + c$

$= \boxed{\ln|\sec x| + c}$

④ $\int (1 + \tan x)^2 dx = \int (1 + 2\tan x + \tan^2 x) dx$

$= \int (1 + \tan^2 x) dx + 2\int \tan x \, dx$

$= \tan x + 2 \ln|\sec x| + c$

⑤ $\int \sec x \, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx$

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$

$= \int \frac{d(\sec x + \tan x)}{\sec x + \tan x}$

$= \ln|\sec x + \tan x| + c.$

⑥ $\int \frac{\sin x + \cos x}{\cos x} dx = \int (\tan x + 1) dx = \ln|\sec x| + x + C$

⑦ $\int \frac{\sec\sqrt{x}}{\sqrt{x}} dx = 2\int \sec x^{\frac{1}{2}} \cdot \frac{1}{2\sqrt{x}} = 2\ln|\sec\sqrt{x} + \tan\sqrt{x}| + C$

⑧ $\int \frac{\sin y}{\cos^2 y} dy = \int \tan y \sec y \, dy = \sec y + c$

(9) $\int (\tan 2x + \sec 2x)^2 dx = \int (\tan^2 2x + 2\tan 2x \cdot \sec 2x + \sec^2 2x)\, dx$

$= \int (\sec^2 2x - 1 + 2\tan 2x \cdot \sec 2x + \sec^2 2x)\, dx$

$= \int (2\sec^2 2x + 2\tan 2x \sec 2x - 1)\, dx$

$= \tan 2x + \sec 2x - x + C.$

(10) $\int \csc\theta\, d\theta = \int \frac{d\theta}{\sin\theta} = \int \frac{d\theta}{\sin 2(\frac{1}{2}\theta)} = \int \frac{d\theta}{2\sin\frac{1}{2}\theta \cdot \cos\frac{1}{2}\theta} \cdot \frac{\cos^2\frac{1}{2}\theta}{\cos^2\frac{1}{2}\theta}$

$= \int \frac{\frac{1}{\cos^2\frac{1}{2}\theta}\,(\frac{1}{2})\, d\theta}{\tan\frac{1}{2}\theta} = \int \frac{(\sec^2\frac{1}{2}\theta)\,\frac{1}{2}\, d\theta}{\tan\frac{1}{2}\theta}.$

$= \int \frac{d(\tan\frac{1}{2}\theta)}{\tan\frac{1}{2}\theta} = \ln|\tan\frac{1}{2}\theta| + C.$

(11) $\int \frac{dx}{1+\cos x} = \int \frac{dx}{1+\cos x} \cdot \frac{1-\cos x}{1-\cos x}$

$= \int \frac{1-\cos x}{1-\cos^2 x}\, dx$

$= \int \frac{1-\cos x}{\sin^2 x}\, dx$

$= \int \csc^2 x - \cot x \csc x \; dx$

$= \csc x - \cot x + C$

(12) $\int \frac{dx}{\csc 2x - \cot 2x} = \int \frac{dx}{\frac{1}{\sin 2x} - \frac{\cos 2x}{\sin 2x}} = \int \frac{dx}{\frac{1-\cos 2x}{\sin 2x}}$

$= \frac{1}{2}\int \frac{\sin 2x\,(2)}{1-\cos 2x}\, dx$

$= \frac{1}{2}\ln|1-\cos 2x| + C$

definite integral

* The Fundemental theorem of Calculus:

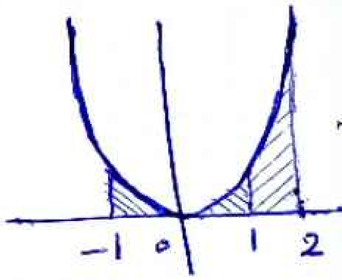
$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

where $F(x)$ is the antiderivative of $f(x)$.
a is the lower bound of the integral.
b is the upper bound of the integral.

Example: Evaluate the definite integral over the interval.

1. $\int_0^1 x^2\,dx$.

solution: $\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{(1)^3}{3} - \frac{(0)^3}{3} = \frac{1}{3}$



2. $\int_{-1}^0 x^2\,dx = \frac{x^3}{3}\Big|_{-1}^0 = \frac{(0)^3}{3} - \frac{(-1)^3}{3} = \frac{1}{3}$

3. $\int_1^2 x^2\,dX = \frac{x^3}{3}\Big|_1^2 = \frac{(2)^3}{3} - \frac{(1)^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$

لايجب اضافة ثابت التكامل في حساب التكامل المحدد.

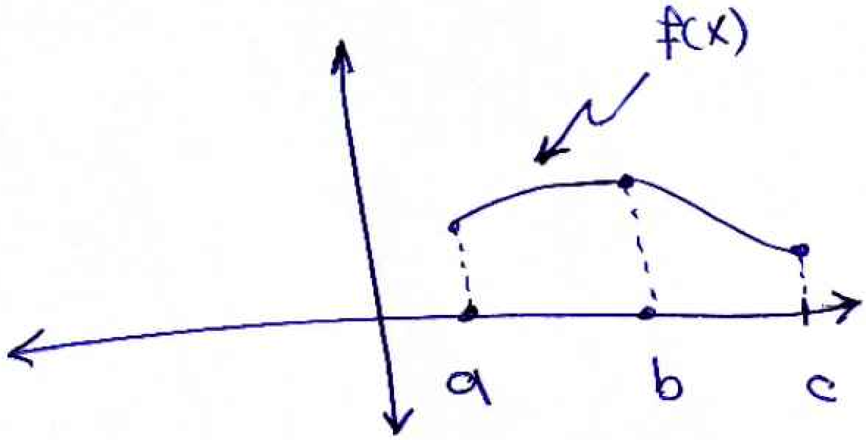
Further Properties:-

① $\int_a^a f(x)\,dx = 0$

② $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

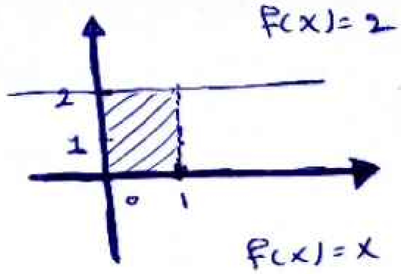
③ $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$.

if $a < b < c$.

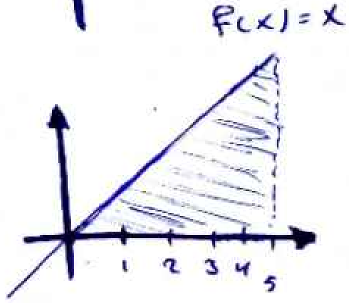


Examples:

① $\int_0^1 2\,dx = 2x\Big|_0^1 = 2(1-0) = 2$



② $\int_0^5 x\,dx = \frac{x^2}{2}\Big|_0^5 = \frac{(5)^2}{2} - \frac{(0)^2}{2} = \frac{25}{2} = 12.5$



③ Find the area under the curve $y = \frac{1}{x}$ on the interval $[2, 4]$.

Sol.
$$A = \int_2^4 \frac{1}{x}\,dx = \ln x\Big|_2^4 = \ln 4 - \ln 2$$
$$= \ln 2^2 - \ln 2$$
$$= 2\ln 2 - \ln 2$$
$$= \ln(2).$$

④ Find the area under the curve $y = e^{-\frac{x}{2}}$ on the interval $[-2, 3]$.

Sol.
$$\int_{-2}^{3} e^{-\frac{x}{2}}\,dx = -2\int_{-2}^{3}\left(-\tfrac{1}{2}\right)e^{-\frac{x}{2}}\,dx$$
$$= -2\, e^{-\frac{x}{2}}\Big|_{-2}^{3}$$
$$= -2\left[e^{-\frac{3}{2}} - e\right] \qquad ; \quad e^{-\frac{3}{2}} = \frac{1}{\sqrt{e^3}}$$
$$= 4.99$$

Examples: - Evaluate the definit integral.

① $\int_1^3 x^3\,dx = \frac{x^4}{4}\Big|_1^3 = \frac{(3)^4}{4} - \frac{(1)^4}{4}$

$= \frac{81}{4} - \frac{1}{4} = \frac{80}{4}$

$= 20$

② $\int_0^2 (-0.1715\ x^4 + 3\sin x + e^{-2x})\,dx$

$= -0.1715\int_0^2 x^4\,dx + 3\int_0^2 \sin x\,dx + \int_0^2 e^{-2x}\,dx$

$= -0.1715\int_0^2 x^4\,dx + 3\int_0^2 \sin x\,dx + \frac{-1}{2}\int_0^2 (-2)\,e^{-2x}\,dx$

$= -0.1715\left(\frac{x^5}{5}\right)\Big|_0^2 - 3(\cos x)\Big|_0^2 - \frac{1}{2}\left(e^{-2x}\right)\Big|_0^2$

$= -0.1715\left[\left(\frac{2^5}{5}\right) - \left(\frac{0^5}{5}\right)\right] - 3\left[\cos(2) - \cos(0)\right] - \frac{1}{2}\left[e^{-2(2)} - e^{-2(0)}\right]$

$= -0.1715\left[\frac{32}{5} - 0\right] - 3\left[-0.4161 - 1\right] - \frac{1}{2}\left[0.0183 - 1\right]$

$= -0.1715(6.4) - 3(-1.4161) - 0.5\left[-0.9817\right]$

$= 3.64155$

Example:- Evaluate the definite integral of the function

$f(x) = 3\sqrt{x} - 2x + 1$ on the interval $[1, 4]$.

Sol. $\int_1^4 f(x)dx = \int_1^4 (3\sqrt{x} - 2x + 1)\,dx = \int_1^4 (3x^{\frac{1}{2}} - 2x + 1)\,dx$

$= \left(3\,\frac{2}{3}\,x^{\frac{3}{2}} - 2\,\frac{x^2}{2} + x\right)\Big|_1^4 = \left(2\sqrt{x^3} - x^2 + x\right)\Big|_1^4$

$= \left[2\sqrt{4^3} - (4)^2 + 4\right] - \left[2\sqrt{(1)^3} - (1)^2 + 1\right]$

$= [2 \times 8 - 16 + 4] - [2 - \cancel{1} + \cancel{1}]$

$= 4 - 2$

$= 2.$

Example:- Evaluate the integral of the function $\left(\frac{3}{x^2} - 1\right)$ on the interval. $[1, 2]$.

Sol. $\int_1^2 \left(\frac{3}{x^2} - 1\right) = \int_1^2 (3x^{-2} - 1)\,dx = 3\,\frac{x^{-1}}{-1} - x$

$= \left(-\frac{3}{x} - x\right)\Big|_1^2 = \left(-\frac{3}{2} - 2\right) - \left(-\frac{3}{1} - 1\right)$

$= -\frac{3}{2} - 2 + 3 + 1$

$= \frac{1}{2}$

Ex:- Evaluate the integral $\int_0^1 (2x-1)^2\,dx$.

Sol.

method ①: Chain Rule

$$\int_0^1 (2x-1)^2\,dx = \frac{1}{2}\int_0^1 2\,(2x-1)^2\,dx$$

$$= \frac{1}{2}\;\frac{(2x-1)^3}{3}\Bigg|_0^1$$

$$= \frac{(2x-1)^3}{6}\Bigg|_0^1$$

$$= \frac{(2(1)-1)^3}{6} - \frac{(2(0)-1)^3}{6}$$

$$= \frac{1}{6} + \frac{1}{6}$$

$$= \frac{1}{3}$$

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method ②:

$$\int_0^1 (2x-1)^2\,dx = \int_0^1 (2x)^2 - 2(2x) + (1)^2\,dx$$

$$= \int_0^1 \left(4x^2 - 4x + 1\right)dx = \frac{4x^3}{3} - 2x^2 + x\,\Bigg|_0^1$$

$$= \left(\frac{4(1)^3}{3} - 2(1)^2 + 1\right) - \cancel{\left(\frac{4(0)^3}{3} - 2(0)^2 + 0\right)}$$

$$= \frac{4}{3} - 2 + 1$$

$$= \frac{1}{3}$$

④

Ex:- Evaluate the definite integral of the function $f(x) = \frac{x-2}{\sqrt{x}}$ on the interval $[1,4]$.

Sol.

$$\int_1^4 \frac{x-2}{\sqrt{x}}\,dx = \int_1^4 \left(\frac{x}{\sqrt{x}} - \frac{2}{\sqrt{x}}\right) dx$$

$$= \int_1^4 \left(x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}\right) dx$$

$$= \left(\frac{2}{3}x^{\frac{3}{2}} - 4x^{\frac{1}{2}}\right)\Big|_1^4$$

$$= \left(\frac{2}{3}\sqrt{x^3} - 4\sqrt{x}\right)\Big|_1^4$$

$$= \left(\frac{2}{3}\sqrt{4^3} - 4\sqrt{4}\right) - \left(\frac{2}{3}\sqrt{(1)^3} - 4\sqrt{1}\right)$$

$$= \frac{2}{3}(8) - 8 - \frac{2}{3} + 4$$

$$= \frac{16}{3} - 8 - \frac{2}{3} + 4$$

$$= \frac{16}{3} - \frac{2}{3} - 4$$

$$= \frac{16}{3} - \frac{2}{3} - \frac{12}{3}$$

$$= \frac{16-2-12}{3}$$

$$= \frac{2}{3}$$

③ $$\int_4^8 \frac{x}{\sqrt{x^2-15}}\,dx = \int_4^8 (x^2-15)^{-\frac{1}{2}}\, x \, dx$$

$$= \frac{1}{2}\int_4^8 \underbrace{(2)\, x}_{\text{مشتقة داخل القوس}} \; \underbrace{(x^2-15)}_{\text{دالة داخل القوس}}^{-\frac{1}{2}} \, dx \quad \text{--- Power Rule.}$$

$$= \cancel{\left(\frac{1}{2}\right)} \frac{(x^2-15)^{\frac{1}{2}}}{\cancel{\left(\frac{1}{2}\right)}} \Bigg|_4^8$$

$$= \sqrt{x^2-15}\;\Big|_4^8$$

$$= \sqrt{(8)^2-15} - \sqrt{(4)^2-15}$$

$$= \sqrt{64-15} - \sqrt{16-15}$$

$$= \sqrt{49} - \sqrt{1}$$

$$= 7 - 1$$

$$= 6.$$

Remarke:
The definit integral can be positive (as in example ① , ② and ③) or Negative or Zero (as in the following examples).

④ $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \sin^2 x \cos x \, dx = \int_{\frac{\pi}{2}}^{\pi} \underbrace{(\sin x)^2}_{\text{الدالة}} \underbrace{\cos x \, dx}_{\text{مشتقتها}}$

$$= \frac{\sin^3 x}{3} \Big|_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{(\sin \pi)^3}{3} - \frac{\left(\sin\left(\frac{\pi}{2}\right)\right)^3}{3}$$

$$= \frac{(0)^3}{3} - \frac{(1)^3}{3}$$

$$= -\frac{1}{3}$$

⑤ $\displaystyle\int_0^{2\pi} \sin x \, dx = -\cos x \Big|_0^{2\pi} = -(\cos 2\pi - \cos(0))$

$$= -(1-1)$$

$$= 0$$

⑥ $\displaystyle\int_1^3 (x^2 - 4x + 3) \, dx = \frac{x^3}{3} - 2x^2 + 3x \Big|_1^3$

$$= \left(\frac{(3)^3}{3} - 2(3)^2 + 3(3)\right) - \left(\frac{1}{3} - 2(1)^2 + 3(1)\right)$$

$$= (\cancel{9} - \cancel{18} + \cancel{9}) - \left(\frac{1}{3} - \cancel{2} + \cancel{3}_{1}\right)$$

$$= -\frac{4}{3}$$

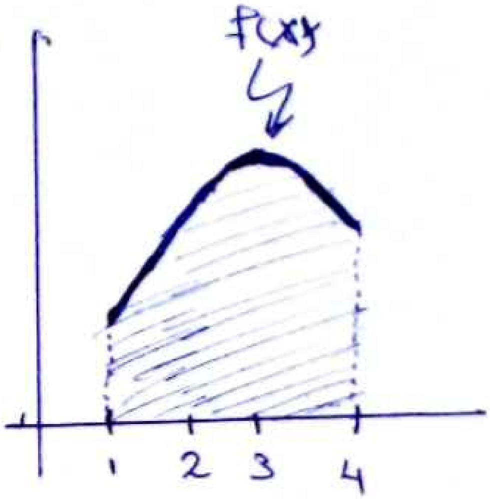
The Area:-

The area under the curve $f(x)$ at the interval $I=[a,b]$ is the definit integral if $f(x)$ is positive at I.

Q: Find the area under the curve

$$f(x) = -0.5x^2 + 3x - 1.5$$

on the interval $[1,4]$.



solution.

$$A = \int_a^b f(x)\,dx$$

$$= \int_1^4 (-0.5x^2 + 3x - 1.5)\,dx$$

$$= \left(-0.5 \cdot \frac{x^3}{3} + 3\,\frac{x^2}{2} - 1.5x\right)\Big|_1^4$$

$$= -\frac{1}{6}x^3 + \frac{3}{2}x^2 - \frac{3}{2}x\,\Big)\Big|_1^4$$

$$= \left[-\frac{1}{6}(4)^3 + \frac{3}{2}(4)^2 - \frac{3}{2}(4)\right] - \left[-\frac{1}{6}(1)^3 + \frac{3}{2}(1)^2 - \frac{3}{2}(1)\right]$$

$$= -\frac{64}{6} + \frac{48}{2} - \frac{12}{2} + \frac{1}{6} \cancel{- \frac{3}{2}} \cancel{+ \frac{3}{2}}$$

$$= -\frac{32}{3} + 24 - 6 + \frac{1}{6}$$

$= 7.5$ area unit

Q: Find the area under the curve $f(x) = x\sqrt{4-x}$ on the interval $[0,4]$.

Sol. $A = \int_0^4 x\sqrt{4-x}\, dx$

when: $x = \boxed{0} \rightarrow u = 4-0 = \boxed{4}$

when: $x = \boxed{4} \rightarrow u = 4-4 = \boxed{0}$

Let $u = 4-x \rightarrow x = 4-u$

and $dx = -du$

$\therefore A = \int_0^4 (4-u)\sqrt{u}\,(-du)$

$= \int_4^0 -4\sqrt{u} + u\sqrt{u}\, du$

$= \int_4^0 -4u^{\frac{1}{2}} + u^{\frac{3}{2}}\, du$

$= \left((-4)\left(\frac{2}{3}\right) \cdot u^{\frac{3}{2}} + \frac{2}{5} u^{\frac{5}{2}}\right)\Big|_4^0$

$= -\frac{8}{3}\sqrt{(4-x)^3} + \frac{2}{5}\sqrt{(4-x)^5}\Big|_0^4$

$= \left(-\frac{8}{3}\sqrt{(0)^3} + \frac{2}{5}\sqrt{(0)^5}\right) - \left(-\frac{8}{3}\sqrt{(4-0)^3} + \frac{2}{5}\sqrt{(4-0)^5}\right)$

$= 0 - \left(-\frac{8}{3}(8) + \frac{2}{5}(32)\right)$

$= \frac{64}{3} - \frac{64}{5}$

$= \frac{5(64) - 3(64)}{15}$

$= \frac{2(64)}{15}$

$= \frac{128}{15}$

$= 8.5\underline{3}$

$8.5\underline{3} = 8.5333333\ldots$ etc.

ملاحظة:

Q1 Find the area under the curve $f(x) = \dfrac{x}{\sqrt{4-x}}$ on the interval $[0, 3]$.

Sol. $A = \int_0^3 \dfrac{x}{\sqrt{4-x}}\,dx$

Let $u = 4 - x \rightarrow x = 4 - u$

$dx = -du$

$\therefore A = \int_0^3 \dfrac{x}{\sqrt{4-x}}\,dx = \int_4^1 \dfrac{4-u}{\sqrt{u}}(-du)$

$= \int_4^1 -4u^{-\frac{1}{2}} + u^{\frac{1}{2}}\,du$

$= (-4)(2)u^{\frac{1}{2}} + (\frac{2}{3})u^{\frac{3}{2}} \Big|_4^1$

$= -8\sqrt{4-x} + \frac{2}{3}\sqrt{(4-x)^3} \Big|_0^3$

$= \left(-8(1) + \frac{2}{3}(1)\right) - \left(-8(2) + \frac{2}{3}(8)\right)$

$= -8 + \frac{2}{3} + 16 - \frac{16}{3}$

$= \dfrac{-24 + 2 + 48 - 16}{3}$

$= \dfrac{10}{3}$

$= 3.\underline{3}$

when: $x = \boxed{0} \rightarrow u = 4 - 0 = \boxed{4}$

when: $x = \boxed{3} \rightarrow u = 4 - 3 = \boxed{1}$

Q: Find the intersection points between the functions $f(x) = x\sqrt{4-x}$ and $g(x) = \frac{x}{\sqrt{4-x}}$.

Sol. the intersection point is the values of $f(x) = g(x)$

$\therefore \quad x\sqrt{4-x} = \frac{x}{\sqrt{4-x}} \quad) * \sqrt{4-x}$

$\Rightarrow \quad x(4-x) = x$

$\Rightarrow \quad 4x - x^2 = x$

$\Rightarrow \quad 4x - x^2 - x = 0 \quad) * -1$

$\Rightarrow \quad x^2 - 3x = 0$

$\Rightarrow \quad x(x-3) = 0$

either $x = 0 \rightarrow f(0) = 0$ --- $(0,0)$

or $x = 3 \rightarrow f(3) = 3\sqrt{4-3} = 3$ --- $(3,3)$

$\therefore$ $(0,0)$ and $(3,3)$ are the intersection points

* Q: Find the area betwee the curves $f(x) = x\sqrt{4-x}$ and $g(x) = \frac{x}{\sqrt{4-x}}$

H.W.

Sol.

Q: Find the maximum point of the function $f(x) = x\sqrt{4-x}$

Ans. $\boxed{\left(\frac{8}{3}, \frac{16}{3\sqrt{3}}\right)}$

Q: Find the area under the curve $\frac{1}{1-\sin x}$ on the interval $[0, \frac{\pi}{3}]$.

Sol.
$$\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin x}\, dx = \int_0^{\frac{\pi}{3}} \frac{1}{1-\sin x} \cdot \left(\frac{1+\sin x}{1+\sin x}\right) dx$$

$$= \int_0^{\frac{\pi}{3}} \frac{1+\sin x}{1-\sin^2 x}\, dx \quad \cdots \quad \sin^2 x + \cos^2 x = 1$$

$$= \int_0^{\frac{\pi}{3}} \frac{1+\sin x}{\cos^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{3}} \left(\frac{1}{\cos x}\right)^2 + \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}\, dx \quad \cdots \begin{cases} \sec x = \frac{1}{\cos x} \\ \tan x = \frac{\sin x}{\cos x} \end{cases}$$

$$= \int_0^{\frac{\pi}{3}} \sec^2 x + \tan x \cdot \sec x\, dx$$

$$= \tan x + \sec x \Big|_0^{\frac{\pi}{3}}$$

$$= \left(\tan\left(\frac{\pi}{3}\right) + \sec \frac{\pi}{3}\right) - \left(\tan(0) + \sec(0)\right)$$

$$= (\sqrt{3} + 2) - (0+1)$$

$$= \sqrt{3} + 1$$

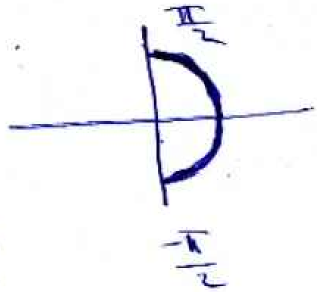
$\approx$ 2.732 Area unit.

Q: Evaluate the integral $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin 2x \, dx$

Sol. we should check when $\sin 2x = 0$.

$\sin 2x = 0$

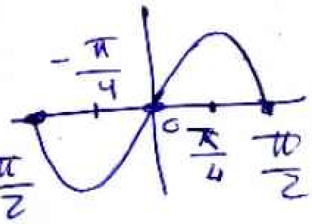
$\rightarrow 2 \sin x \cdot \cos x = 0$



either $\sin x = 0 \rightarrow x = 0$

or $\cos x = 0 \rightarrow x = \frac{\pi}{2}, -\frac{\pi}{2}$

so that we shoul devide the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ into $[-\frac{\pi}{2}, 0]$ and $[0, \frac{\pi}{2}]$



$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin 2x \, dx = \int_{-\frac{\pi}{2}}^{0} \sin 2x \, dx + \int_{0}^{\frac{\pi}{2}} \sin 2x \, dx$$

$$= \left(-\frac{1}{2} \cos 2x\right)\Big|_{-\frac{\pi}{2}}^{0} + \left(-\frac{1}{2} \cos 2x\right)\Big|_{0}^{\frac{\pi}{2}}$$

$$= \left(-\frac{1}{2}\right)\left(\cos(0) - \cos 2\left(-\frac{\pi}{2}\right)\right) + \left(-\frac{1}{2}\right)\left(\cos 2\left(\frac{\pi}{2}\right) - \cos 0\right)$$

$$= -\frac{1}{2}(1 - (-1)) + \left(-\frac{1}{2}\right)(-1 - 1)$$

$$= -1 + 1$$

$$= 0$$

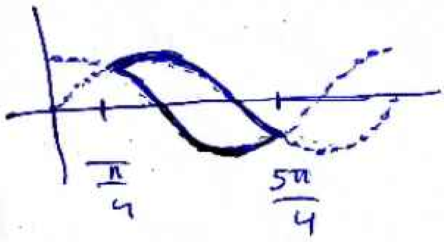
Note; If we need the Area then we have to take the absolute value of each integral

$A = |-1| + |1| = 2$

Q: Find the area between $f(x) = \sin x$ and $g(x) = \cos x$ on the interval

Sol.

$$A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)$$



$$= (-\cos x - \sin x) \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= -\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4} - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right)$$

$$= -\left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{4}{\sqrt{2}}$$

$= 2\sqrt{2}$. Area unit.

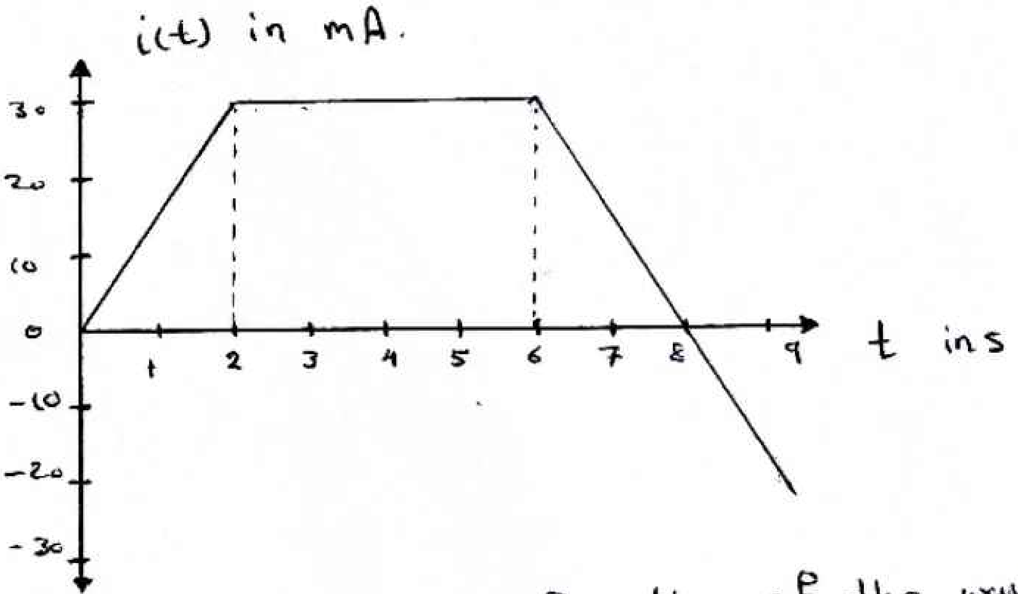
Q: Find the intersection points between the curves $f_1(x) = \sin x$, $f_2(x) = \cos x$ on the interval $[0, 3\frac{\pi}{2}]$.

Ans: $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}\right)$.

Example: For the waveform of current $i(t)$ shown in fig(1) compute the total charge $q(t)$ transfered between

1, $0 \le t \le 3$ s

2, $0 \le t \le 9$ s.



Sol. first we have to find the picewise function of the wave-form $i(t)$; s.t.

$$i(t) = \begin{cases} 15t & 0 \le t \le 2 \\ 30 & 2 \le t \le 6 \\ -15t+120 & 6 \le t \le 9 \end{cases}$$

① $q(t) = \int_{t_0}^{t} i(t)\,dt + q(t_0)$; where $q(t_0)=0$.

$= \int_0^3 i(t)\,dt$

$= \int_0^2 i(t)\,dt + \int_2^3 i(t)\,dt$

$= \int_0^2 15t\,dt + \int_2^3 30\,dt$

$= 15\frac{t^2}{2}\Big|_0^2 + 30t\Big|_2^3$

$= \left(15\frac{(2)^2}{2} - \cancel{15\frac{(0)^2}{2}}\right) + \left(30(3) - 30(2)\right)$

$= 30 + 30$

$= 60$ mC.

2, $0 \le t \le 9\,s$

Sol. $q(t) = \int_{t_0}^{t} i(t)\,dt + q(t_0)$

$$= \int_{0}^{9} i(t)\,dt$$

$$= \int_{0}^{2} i(t)\,dt + \int_{2}^{6} i(t)\,dt + \int_{6}^{9} i(t)\,dt$$

$$= \int_{0}^{2} (15t)\,dt + \int_{2}^{6} (30)\,dt + \int_{6}^{9} (-15t+120)\,dt$$

$$= \frac{15t^2}{2}\Big|_0^2 + 30t\Big|_2^6 + \left(-\frac{15t^2}{2} + 120t\right)\Big|_6^9$$

$$= \left(\frac{15(2)^2}{2} - \frac{15(0)^2}{2}\right) + \left(30(6) - 30(2)\right) + \left(-\frac{15(9)^2}{2} + \frac{15(6)^2}{2} + 120(9) - (120)(6)\right)$$

$$= (30-0) + (30(4)) + \left(-\frac{15}{2}(81-36) + 120(9-6)\right)$$

$$= 30 + 120 - \frac{(15)(45)}{2} + 120(3)$$

$$= 30 + 120 - 337.5 + 360$$

$$= 172.5\ mC.$$

Note For the picewise function $i(t)$:
the first function we use $m = \frac{\Delta y}{\Delta x} = \frac{30-0}{2-0} = 15$

$\rightarrow m = \frac{y-y_0}{x-x_0} \rightarrow 15 = \frac{y-30}{x-2} \rightarrow y-30 = 15x - 30 \rightarrow \boxed{y = 15x}$

or $i(t) = 15t$.

For the third function $m = \frac{\Delta y}{\Delta x} = \frac{30-0}{6-8} = -15$

$\rightarrow m = \frac{y-y_0}{x-x_0} \rightarrow -15 = \frac{y-30}{x-6} \rightarrow y-30 = -15x + 90$

$\rightarrow y = -15x + 120$ or $i(t) = -15t + 120$

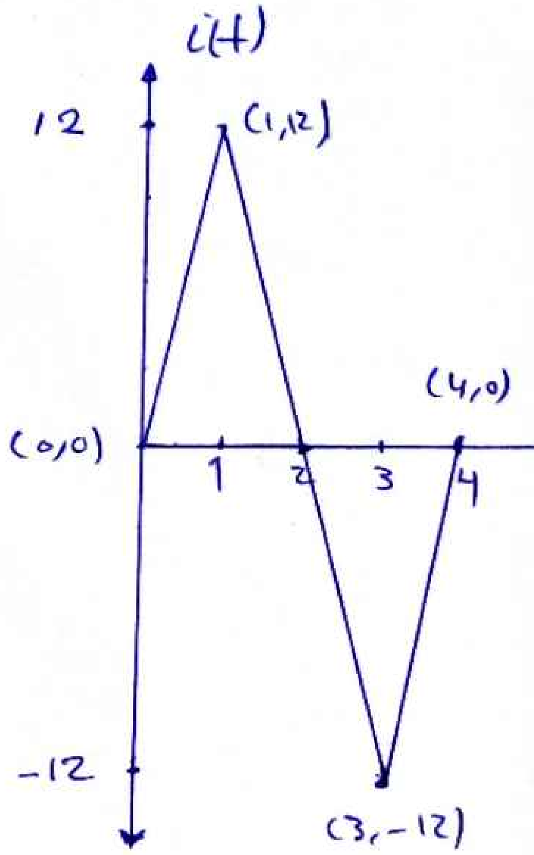
Q: Consider the wave form shown figure.

① find the pice wise function form.

② Compute the total charge on time interval

ⓐ $0 \le t \le 4$

ⓑ $0 \le t \le 3$



Sol.

for the 1st pice :- use $(0,0)$ & $(1,12)$

$$m = \frac{\Delta i}{\Delta t} = \frac{12-0}{1-0} = 12.$$

$$12 = \frac{i-0}{t-0} \Rightarrow i = 12t. \quad \text{for } 0 \le t \le 1$$

for the 2nd pice :- use $(1,12)$ & $(3,-12)$.

$$m = \frac{\Delta i}{\Delta t} = \frac{12-(-12)}{1-3} = \frac{24}{-2} = -12.$$

$$-12 = \frac{i-12}{t-1} \Rightarrow i-12 = -12(t-1)$$

$$i-12 = -12t+12. \Rightarrow i = -12t+24 \; ; \text{for } 1 \le t \le 3$$

for the 3rd pice, use $(3,-12)$ & $(4,0)$

$$m = \frac{\Delta i}{\Delta t} = \frac{0-(-12)}{4-3} = 12$$

$$12 = \frac{i-0}{t-4} \Rightarrow i-0 = 12(t-4) \Rightarrow i = 12t-48 \quad \text{for } 3 \le t \le 4$$

$\therefore$ the picewise function is.

$$i(t) = \begin{cases} 12t & 0 \le t \le 1 \\ -12t + 24 & 1 \le t \le 3 \\ 12t - 48 & 3 \le t \le 4 \end{cases}$$

---

(2) (a) $q(t) = \int_0^4 i(t)\,dt$

$$= \int_0^1 i(t)\,dt + \int_1^3 i(t)\,dt + \int_3^4 i(t)\,dt$$

$$= \int_0^1 (12t)\,dt + \int_1^3 (-12t + 24)\,dt + \int_3^4 (12t - 48)\,dt$$

$$= 6t^2 \Big|_0^1 + (-6t^2 + 24t)\Big|_1^3 + (6t^2 - 48t)\Big|_3^4$$

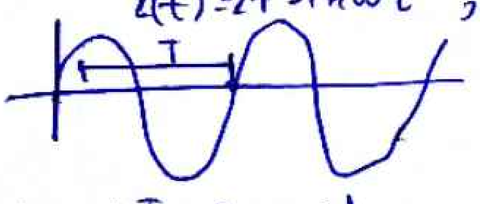
$$= 6(1-0) + (-6(3)^2 + 24(3) - (-6(1)^2 + 24(1))) + 6((4))^2 - 48(4) - (6(3)^2 - 48(3))$$

$$= 68 \text{ mC}$$

(b) H.w.

Ex: Find $q(t)$ where $i(t)$ defined in the figure

$i(t) = A \sin \omega t$; $\omega = \frac{2\pi}{T}$



sol: $q(t) = \int_0^{T/2} i(t)\,dt$

$$= \int_0^{T/2} A \sin \omega t \, dt$$

$$= A \cdot \left(-\frac{\cos \omega t}{\omega}\right)\Big|_0^{T/2} = -\frac{A}{\omega}\left(\cos \frac{\omega T}{2} - \cos(0)\right)$$

$$= -\frac{A}{\omega}(-1-1) = \frac{2A}{\omega} \text{ mC.}$$

Example: rho

Consider 12-Volte battery source with curent function.

$$i(t) = \begin{cases} 8e^{-\frac{t}{3600}} & 0<t<8 \\ 0 & o.w. \end{cases}$$

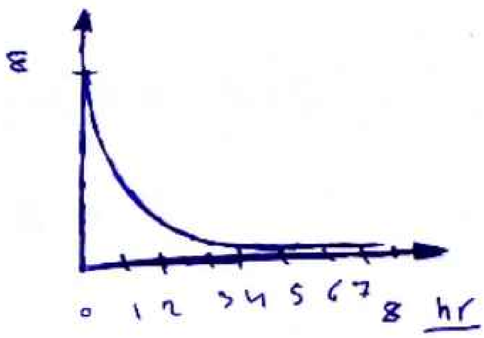
<ins>Find:</ins> ① Total charge $q(t)$

② maximum power P_{max}

③ The total energy $w(t)$

④ The average power. P_{ave}.

<ins>Sol.</ins>



① $q(t) = \int_{t_0}^{t_1} i(t)\,dt$

$= \int_0^{8\times60\times60} 8e^{-\frac{t}{3600}}\,dt$

$= 8(-3600)\int_0^{28800} \left(-\frac{1}{3600}\right) e^{-\frac{t}{3600}}\,dt$

$= -28800\left[e^{-\frac{t}{3600}}\right]_0^{28800}$

$= -28800\left[e^{-8} - 1\right]$

$= 28790$ C.

② $P_{max} = V_{max} \cdot i_{max}$

$= (12)(8)$

$= 96$ wat.

③ $w(t) = \int_{t_0}^{t_1} p(t)\,dt = \int_{t_0}^{t_1} v.i\,dt$

$= \int_0^{8\times60\times60} (12)(8e^{-\frac{t}{3600}})\,dt$

$= (-96)(3600)\left[e^{-\frac{t}{3600}}\right]_0^{28800}$

$= 345484.064$ J.

④ $P_{ave} = \frac{1}{T} \int_0^T P(t)\,dt$

$$= \frac{1}{28800} \int_0^{28800} 96\, e^{-\frac{t}{3600}}\, dt$$

$$= \frac{345484.664}{28800} = 11.996 \text{ wat.}$$

The Trigonometric integrals.

$\int \sin^2 x \, dx$ And $\int \cos^2 x \, dx$.

① $\int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x) \, dx$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$$

$$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x \, dx$$

$$= \frac{1}{2}x - \left(\frac{1}{2}\right) \cdot \frac{1}{2}\sin 2x + c$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c.$$

② $\int \cos^2 x \, dx = \int \frac{1}{2}(1 + \cos 2x) \, dx$

$$= \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx$$

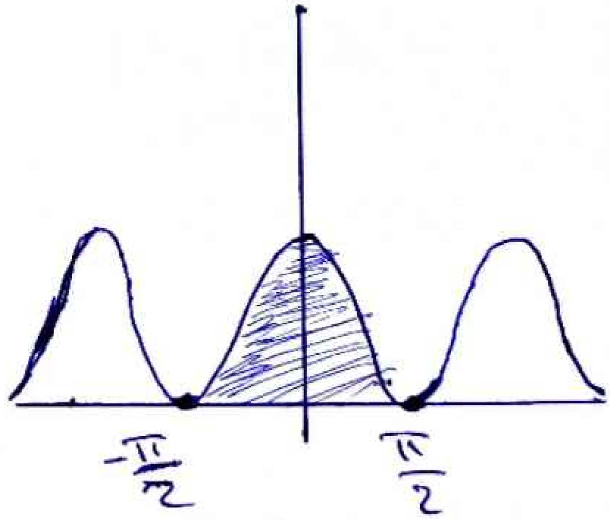
$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x \, dx$$

$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

Q2: Find the area under the curve

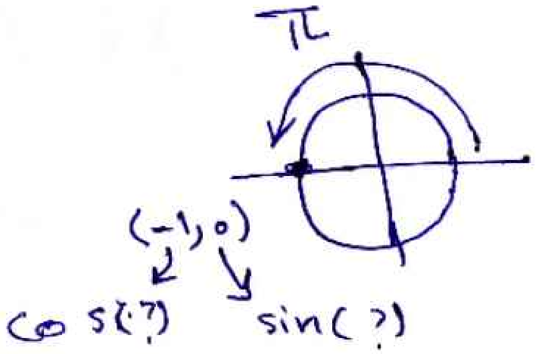
$F(x) = \cos^2 x$ at the interval

$[-\frac{\pi}{2}, \frac{\pi}{2}]$



solution:

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x \, dx$$



$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2x) \, dx$$

$$= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos 2x) \, dx$$

$$= \frac{1}{2} \left[x + \frac{1}{2} \sin 2x \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \left[\left(\frac{\pi}{2} + \underbrace{\frac{1}{2} \sin\left(2 \frac{\pi}{2}\right)}_{0} \right) \underset{+}{-} \left(-\frac{\pi}{2} + \underbrace{\frac{1}{2} \sin\left(2\left(-\frac{\pi}{2}\right)\right)}_{0} \right) \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{2} + \frac{\pi}{2} \right]$$

$$= \frac{\pi}{2}.$$ area unit

③ $$\int \sin^2 x \cdot \cos x \, dx = \int (\sin x)^2 \cdot \cos x \, dx$$

دالة داخل القوس   مشتقة داخل القوس

$$= \frac{\sin^3 x}{3} + C.$$

④ $$\int \cos^2 x \cdot \sin x \, dx = \int (\cos x)^2 \sin x \, dx$$

$$= -\int (\cos x)^2 (-\sin x) \, dx$$

دالة داخل القوس   مشتقة داخل القوس

$$= -\frac{\cos^3 x}{3} + C.$$

⑤ $$\int \sin^3 x \cos x \, dx = \int (\sin x)^3 \cdot \cos x \, dx$$

دالة داخل القوس   مشتقة داخل القوس

$$= \frac{\sin^4 x}{4} + C.$$

⑥ $\int \cos^3 x \cdot \sin x \, dx = \int (\cos x)^3 \sin x \, dx$

$$= -\int (\cos x)^3 (-\sin x) \, dx$$

دالة داخل القوس　　　مشتقة داخل القوس

$$= -\frac{\cos^4 x}{4} + C.$$

⋮

and so on

$$\int \sin^m x \cdot \cos x \, dx = \frac{\sin^{m+1} x}{m+1} + C.$$

$$\int \cos^m x \cdot \sin x \, dx = -\frac{\cos^{m+1}}{m+1} + C$$

⑦ $\int \cos x \cdot \sin x = ?$

$\int \cos x \sin x \, dx = -\int (\cos x)^1 (-\sin x) \, dx$

الدالة　　المشتقة

$$= -\frac{\cos^2 x}{2} + C_1$$

$\int \sin x \cos x \, dx = \int (\sin x)^1 . \cos x \, dx$

الدالة　　المشتقة

$$= \frac{\sin^2 x}{2} + C_2$$

الحلين متساويين مع اختلاف الثابتة C_1 و C_2

Q; Prove that $\frac{\sin^2 x}{2} + C_1 = -\frac{\cos^2 x}{2} + C_2$

where $C_2 = C_1 + \frac{1}{2}$.

Sol.: $\cos^2 x + \sin^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$.

$$\frac{\sin^2 x}{2} + C_1 = \frac{1 - \cos^2 x}{2} + C_1$$

$$= -\frac{\cos^2 x}{2} + \left(C_1 + \frac{1}{2}\right), \text{ let } \underline{C_2 = C_1 + \frac{1}{2}}$$

$$= -\frac{\cos^2 x}{2} + C_2.$$