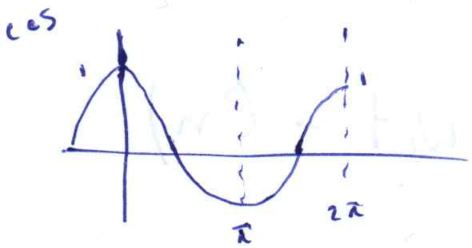
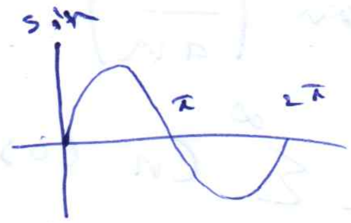
Baron Fourier introduced the idea that any periodic function can be represented by a series of sines & cosines which are harmonically related.

$$
\begin{aligned}
f(t) = {} & a_0 \cos 2\pi(0)t + b_0 \sin 2\pi(0)t \\
& + a_1 \cos 2\pi(f_0)t + b_1 \sin 2\pi(f_0)t \\
& + a_2 \cos 2\pi(2f_0)t + b_2 \sin 2\pi(2f_0)t \\
& + a_3 \cos 2\pi(3f_0)t + b_3 \sin 2\pi(3f_0)t \\
& \vdots
\end{aligned}
$$

Frequency	coefficient of cos component	coefficient of sin component	cos component	sin component
0	a_0	b_0	$a_0 \cos \cancel{2\pi(0)t}$	$b_0 \cancel{\sin 2\pi(0)} t$
f_0	a_1	b_1	$a_1 \cos 2\pi f_0 t$	$b_1 \sin 2\pi f_0 t$
$2f_0$	a_2	b_2	$a_2 \cos 2\pi 2 f_0 t$	$b_2 \sin 2\pi 2 f_0 t$
$3f_0$	a_3	b_3	$a_3 \cos 2\pi 3 f_0 t$	$b_3 \sin 2\pi 3 f_0 t$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	a_∞	b_∞	$a_\infty \cos 2\pi \infty f_0 t$	$b_\infty \sin 2\pi \infty f_0 t$

$$\therefore \quad f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$





Fourier Series :- A function f(t) can be expressed as a series of sines & cosines :-

$$f(t) = \underbrace{\frac{a_0}{2}}_{\text{DC Part}} + \underbrace{\sum_{n=1,2,3}^{\infty} a_n \cos \frac{2\pi n t}{T}}_{\text{Even Part}} + \underbrace{\sum_{n=1,2,3}^{\infty} b_n \sin \frac{2\pi n t}{T}}_{\text{odd Part}}$$

T = Period of the Signal

$w_0 = 2\pi f$

Let $W_0 = \frac{2\pi}{T}$ fundamental angular frequency

$w_n = n w_0$ n-th harmonic

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\underline{n W_0} t) + \sum_{n=1}^{\infty} b_n \sin(n w_0 t)$$

$$a_0 = \frac{2}{T} \int_0^T f(t)\, dt$$ DC part or average magnitude of the signal over one time period.

$$a_n = \frac{2}{T} \int_0^T f(t) \cos n w_0 t \, dt \qquad n = 1, 2, \ldots$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin n w_0 t \, dt \qquad n = 1, 2, \ldots$$

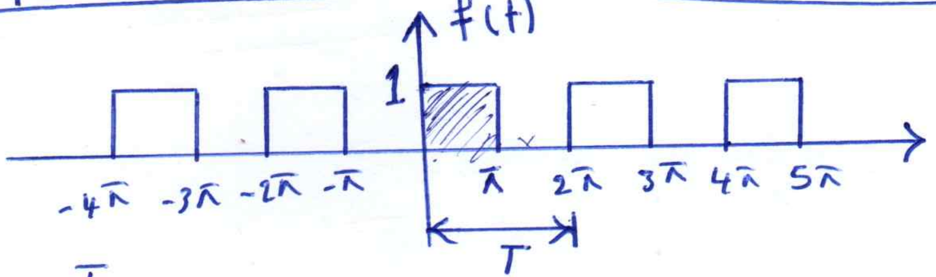
$$c_n = \sqrt{a_n^2 + b_n^2}$$

$$\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

$$f(t) = \frac{a_0}{2} \sum_{n=1}^{\infty} C_n \cos(n W_0 t - \theta_n)$$

ex| Square Wave

$$f(t) = \begin{cases} 0, & -\pi < t \le 0 \\ 1, & 0 < t \le \pi \end{cases}$$



$$\frac{2}{T}\left[\int_0^{\pi} 1\, dt + \int_{\pi}^{2\pi} 0\, dt\right]$$

$T = 2\pi$

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt = \frac{\cancel{2}}{\cancel{2}\pi}\int_0^{\pi} 1\, dt = \frac{1}{\cancel{\pi}}(\cancel{\pi} - 0) = 1 \qquad \boxed{a_0 = 1}$$

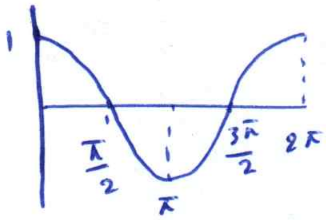
$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\, dt = \frac{\cancel{2}}{\cancel{2}\pi}\int_0^{\pi}\cos nt\, dt = \frac{1}{n\pi}\sin nt\Big|_0^{\pi}$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$\boxed{a_n = 0} \qquad \text{for } n = 1, 2, \ldots$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin nt\, dt = \frac{-1}{n\pi}\cos nt\Big|_0^{\pi}$$

$$= \frac{-1}{n\pi}(\cos n\pi - 1) = \begin{cases} \frac{2}{n\pi} & n = 1, 3, 5 \\ 0 & n = 2, 4, 6 \end{cases}$$



$\cos\pi = \cos 3\pi = \cos 5\pi = -1$

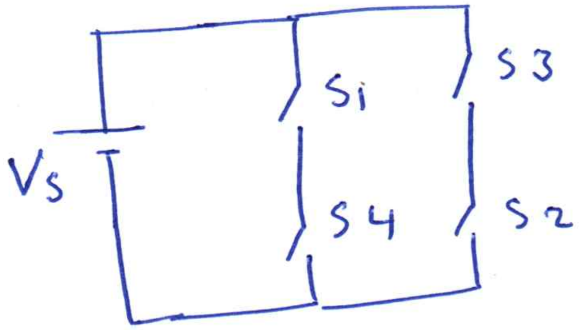
$\cos 2\pi = \cos 4\pi = \cos 6\pi = 1$

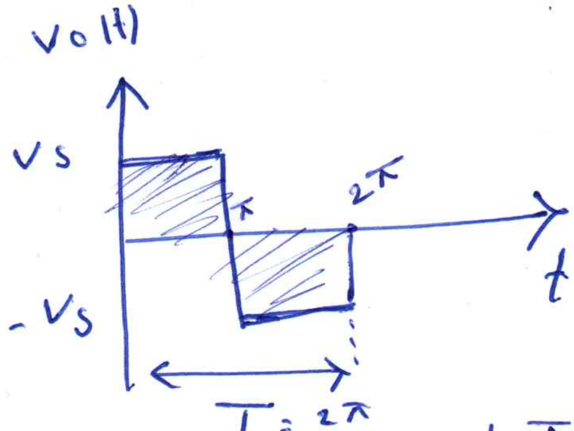
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

$$= \frac{1}{2} + 0 + \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right)$$

o/p Voltage of

ex] Single phase full bridge inverter –





$$f(t) = V_o(t) = \frac{a_o}{2} + \sum_{n=1}^{n=\infty} a_n \cos \frac{2\pi n t}{T} + \sum_{n=1}^{n=\infty} b_n \sin \frac{2\pi n t}{T}$$

$$f(t) \begin{cases} V_s & 0 < t < \pi \\ -V_s & \pi < t < 2\pi \end{cases}$$

$$w_o = \frac{2\pi}{T}$$

$$f(t) = V_o(t) = \frac{a_o}{2} + \sum_{n=1}^{n=\infty} a_n \cos n w_o t + \sum_{n=1}^{n=\infty} b_n \sin n w_o t$$

$$a_o = \text{DC Part} = V_{odc} = \frac{2}{T}\int_0^T V_o(t)\,dt$$

$$= \frac{\cancel{2}1}{\cancel{2}\pi}\left[\int_0^\pi V_s\,dt + \int_\pi^{2\pi} -V_s\,dt\right]$$

$$= \frac{V_s}{\pi}\left[t\Big|_0^\pi - t\Big|_\pi^{2\pi}\right] = \frac{V_s}{\pi}\left[(\pi - 0) - (2\pi - \pi)\right]$$

$$= \text{Zero}$$

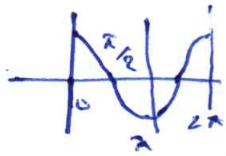
$$a_n = \frac{2}{T}\int_0^T f(t)\cos n w_o t\,dt = \frac{2}{2\pi}\left[\int_0^\pi V_s \cos n w_o t\,dt + \int_\pi^{2\pi} -V_s \cos n w_o t\,dt\right]$$

$$= \frac{V_s}{\pi n w_o}\left[\sin n w_o t\Big|_0^\pi - \sin n w_o t\Big|_\pi^{2\pi}\right]$$

$$= \frac{V_s}{\pi n w_o}\left[(\underbrace{\sin n w_o \pi - \sin n w_o 0}_{(0\ -\ 0)}) - (\underbrace{\sin n w_o 2\pi - \sin n w_o \pi}_{(0\ -\ 0)})\right]$$

$$a_n = \text{Zero}$$

$$b_n = \frac{2}{T}\int_0^T V_o(t)\sin n\omega_0 t \, dt$$



$$= \frac{2}{T}\left[\int_0^{\pi} V_s \sin n\omega_0 t + \int_{\pi}^{2\pi} -V_s \sin n\omega_0 t \, dt\right]$$

$$= \frac{2V_s}{T n \omega_0}\left[-\cos n\omega_0 t \Big|_0^{\pi} + \cos n\omega_0 t \Big|_{\pi}^{2\pi}\right]$$

$$\omega_0 = 2\pi f = \frac{2\pi}{T}$$

$$= \frac{\cancel{2} V_s \cancel{T}}{n \cancel{T} \cancel{2}\pi}\left[-\cos n \frac{2\pi}{T}\frac{T}{2} + \cos 0 + \cos n \frac{2\pi}{\cancel{T}}\cancel{T}\right.$$

$$\left. - \cos n \frac{2\pi}{T}\frac{T}{2}\right]$$

$$= \frac{V_s}{n\pi}\left[1 - 2\cos n\pi + \cos 2n\pi\right]$$

$$b_n = \frac{4V_s}{n\pi} \quad \text{for } n = 1, 3, 5$$

$1 - 2\cos\pi + \cos 2\pi$
$1 - 2(-1) + 1 = 4$

$$b_n = \text{zero} \quad \text{for } n = 2, 4, 6$$

$1 - 2\cos 2\pi + \cos 4\pi$
$1 - 2(1) + 1 = \text{zero}$

$$\therefore \; V_o(t) = \sum_{n=1}^{n=\infty} \frac{4V_s}{n\pi}\sin n\omega_0 t$$

$$V_o(t) = \frac{4V_s}{\pi}\sin\omega_0 t + \frac{4V_s}{3\pi}\sin 3\omega_0 t + \frac{4V_s}{5\pi}\sin 5\omega_0 t$$

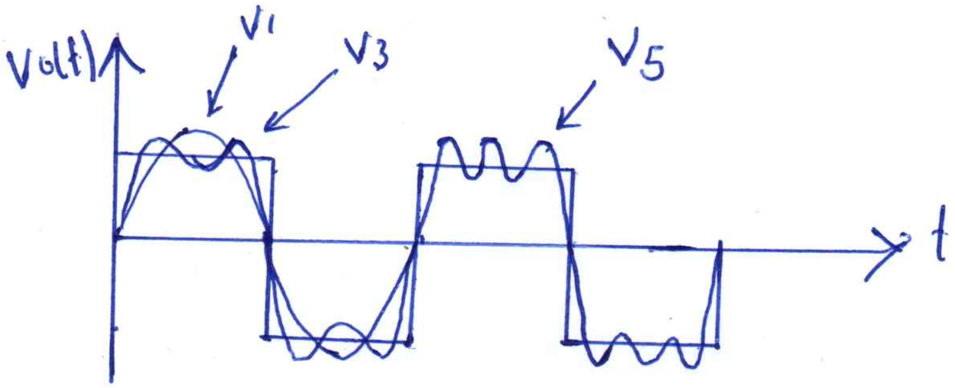
$$V_{1max} = 4V_s/\pi = 0.9V_s \quad (n=1)$$

$$V_o(t) = V_1 + V_3 + V_5 + \cdots$$

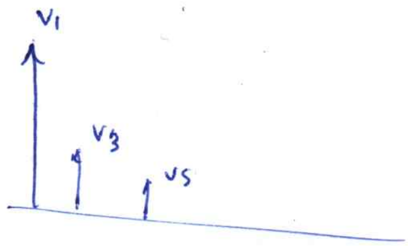
V_1 ↑ fund. comp

V_3 ↑ third harmonic

V_5 ↖ fifth harmonic



ex)

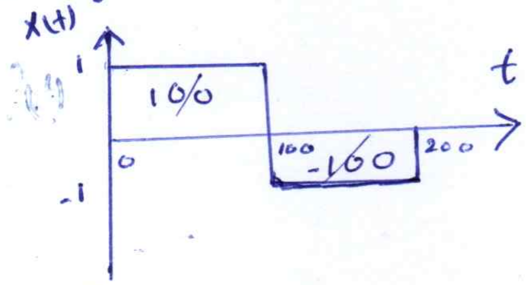


- The coefficients of Fourier Series (a_0, a_n & b_n) are dependent on the Shape or pattern of the Signal.

1- $a_0 = \frac{1}{T}\int_0^T f(t)\,dt$

This coefficient is actually the average magnitude of the signal over one time period (or is the DC component of the signal).

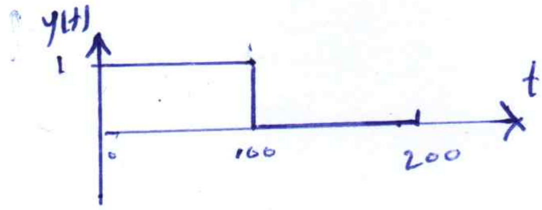
ex)



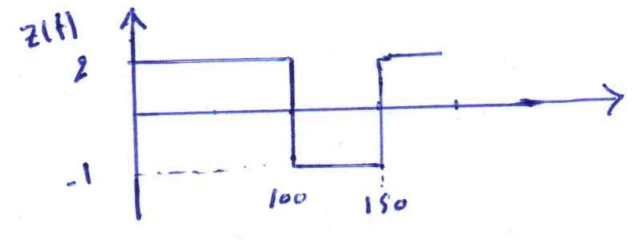
$a_0 = 0$

$a_0 = \frac{1*100 + (-1)*100}{200} = 0$

because areas over & under x-axis over one time are equal.



$a_0 = \frac{100*1 + 0*100}{200} = 0.5$



$a_0 = \frac{2*100 + (-1)*50}{150} = 1$

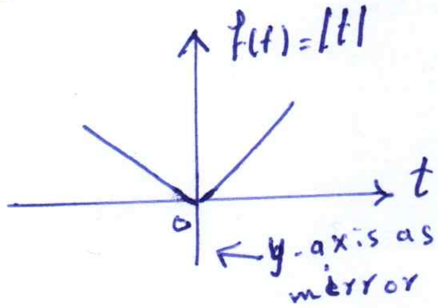
In some cases, the geometric predication or visualization is not so easy, then we need to calculate the areas over and below the x-axis over one time period.

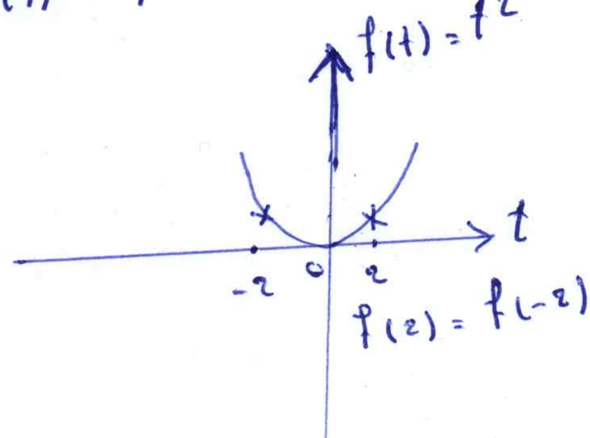
2- an & bn

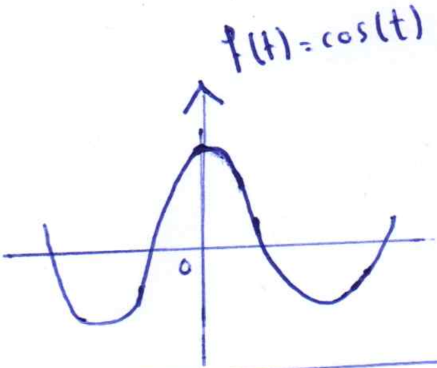
- We can classify signal into three types
  1. odd
  2. even
  3. odd + even

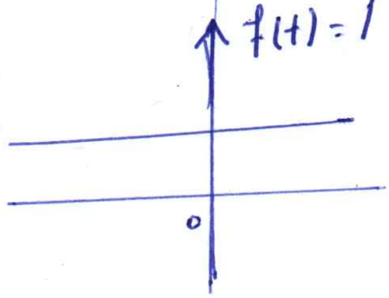
- For odd signals $\longrightarrow a_n = 0$
- In even signal $\longrightarrow b_n = 0$.
- In (even+odd) signal $\longrightarrow a_n \neq 0 , b_n \neq 0$

- Any function $f(t)$ is even if its plot is symmetrical about the vertical axis $f(t) = f(-t)$ as following:-

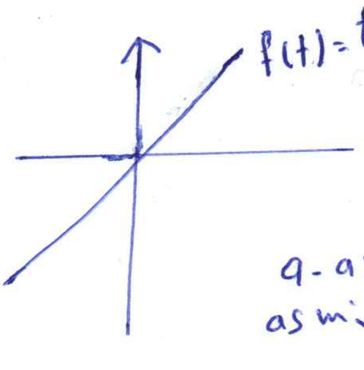


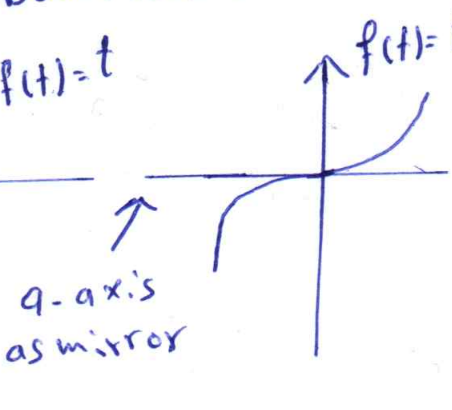


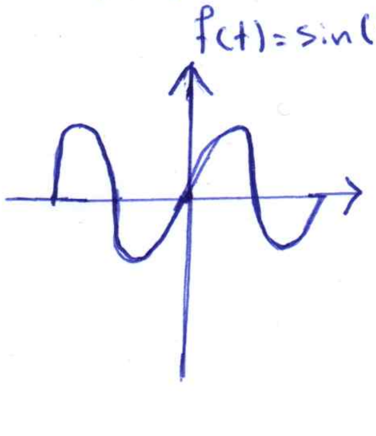


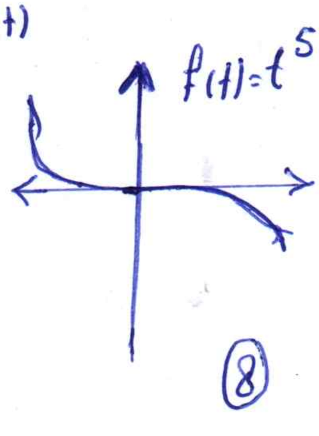


- On the other hand, any function is odd if its anti-symmetrical about the vertical axis. $f(t) = -f(-t)$ as:.

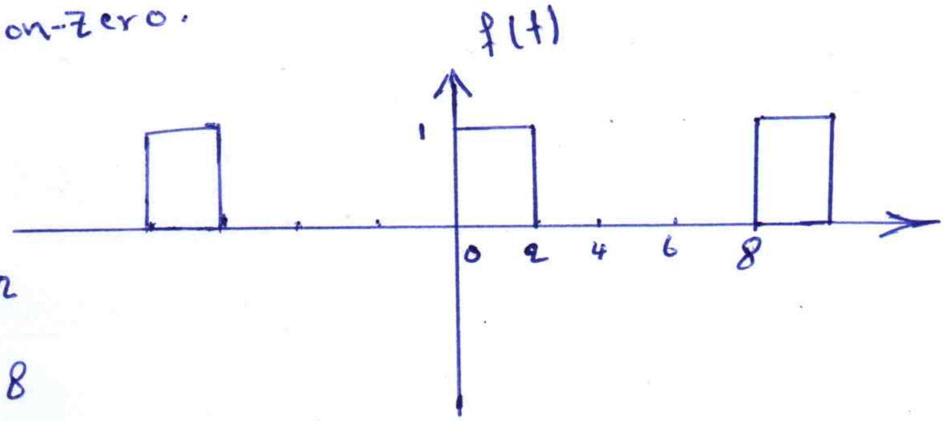








- The following function is neither odd nor even, thus both a_n and b_n are non-zero.



$$\therefore f(t) = \begin{cases} 1 & 0 > t > 2 \\ 0 & 2 > t > 8 \end{cases}$$

H.W

$$a_0 = \frac{(1*2)+(0*6)}{8} = 1/4$$

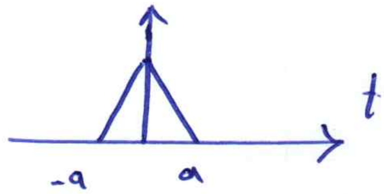
$a_n =$

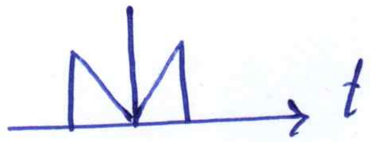
Even and Odd Functions

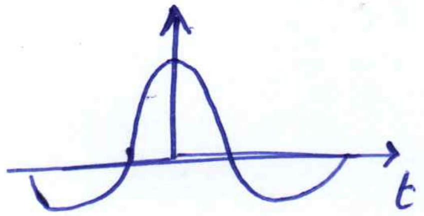
- A function $f(x)$ is even if the graph of $f(x)$ is symmetric with respect to the y-axis.

$$f(-x) = f(x) \qquad \text{for all } x \text{ in the domain of } f(x)$$

ex)







$x(t) = \cos \omega t$

$\cos(-\theta) = \cos\theta$

---

ex) $f(x) = -3x^2 + 4 \longrightarrow f(-x) = -3(-x)^2 + 4$
$= -3x^2 + 4$

$\therefore f(-x) = f(x)$

---

ex) $f(x) = \sin^2 4x \longrightarrow f(-x) = \sin^2(-4x)$

$= [\sin(-4x)]^2$

$= [-\sin(4x)]^2$

$= \sin^2(4x)$

$= f(x)$

- A function $f(x)$ is odd if the graph of $f(x)$ is symmetric with respect to the origin.

$f(-x) = -f(x)$ for all x in the domin of $f(x)$

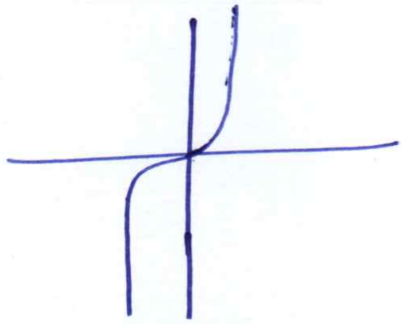
ex) $f(x) = \sin 5x^2 \longrightarrow f(-x) = \sin 5(-x)^2$
$= \sin 5x^2 = f(x)$

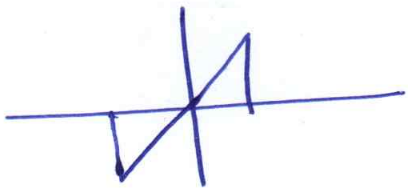
thus this fun is even

ex) $f(x) = \sin 3x \longrightarrow f(-x) = \sin(-3x)$
$= -\sin(3x)$
$= -f(x)$

thus this fun is odd

ex)





Neither an even nor an odd function

ex) $f(x) = x^3 - 5x + 4$

$f(-x) = f(x)$ even

$f(-x) = -f(x)$ odd

$\therefore f(-2) = -2^3 - 5(-2) + 4 = 6$

$f(2) = 2$

$-f(2) = -2$

Thus this function neither an ever nor an odd function

ex)

ex) $f(x) = x \qquad 0 < x < 2\pi$ H.W

Sol) $f(x) = \pi - 2\left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots \right)$

---

ex) Find the fourier series of

$$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

Sol)

0-2pi

ex) $f(x) = \frac{x}{2}$

sol) $a_0 = \frac{2}{T}\int_0^T f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{x}{2}\,dx = \frac{1}{4\pi}\, x^2\Big|_0^{2\pi} = \frac{\cancel{4}\pi^{\cancel{2}}}{\cancel{4\pi}}$

$= \pi$

$a_n = \frac{2}{T}\int_0^T f(x)\cos nx\,dx = \underbrace{\frac{1}{\pi}\int_0^{2\pi} \frac{x}{2}\cos nx\,dx}_{\text{integration by part}}$

$\int u\,dv = uv - \int v\,du$

$u = x \longrightarrow du = dx$

$dv = \cos nx\,dx \longrightarrow v = \frac{\sin nx}{n}$

$a_n = \frac{1}{2\pi}\left[\frac{x}{n}\sin nx\Big|_0^{2\pi} - \frac{1}{n}\int_0^{2\pi}\sin nx\,dx\right] = \frac{x}{n}\sin nx\Big|_0^{2\pi} - \frac{1}{n}\left(\frac{-\cos nx}{n}\right)$

$= \frac{1}{2\pi}\left[\left(\frac{2\pi}{n}\underset{0}{\cancel{\sin n2\pi}} - 0\right) - 0\right] = 0$

$b_n = \frac{2}{T}\int_0^T f(x)\sin nx\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\sin nx\,dx$

$u = x \longrightarrow du = dx$

$dv = \sin nx \longrightarrow v = \frac{-\cos nx}{n}$

$= \frac{1}{2\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right)\right]_0^{2\pi} - \int \frac{-\cos nx}{n}\,dx\right\}$

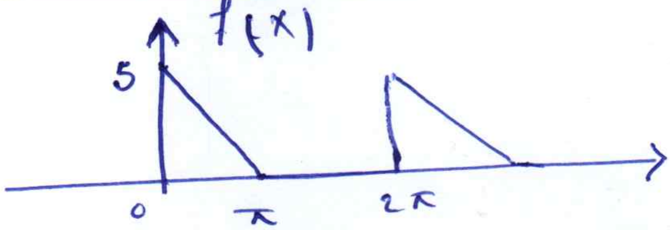
$= \frac{1}{\cancel{2\pi}}\left\{\frac{\cancel{2\pi}}{n}\left(-\underset{-1}{\cancel{\cos n2\pi}}\right) + \cancel{\frac{\sin nx}{n^2}}\Big|_0^{2\pi}\right.$

$= \frac{-1}{n}$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} -\frac{1}{n}\sin nx$$

$$= \frac{\pi}{2} - \left\{ \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots \right\}$$

ex)



$$f(x) \begin{cases} 5-x & , 0 < x < \pi \\ 0 & , \pi < x < 2\pi \end{cases}$$

sol)

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt = \frac{1}{\pi}\int_0^{\pi}(5-x)\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 0\,dx$$

$$= \frac{1}{\pi}\left[5x - \frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(5\pi - \frac{\pi^2}{2}\right)$$

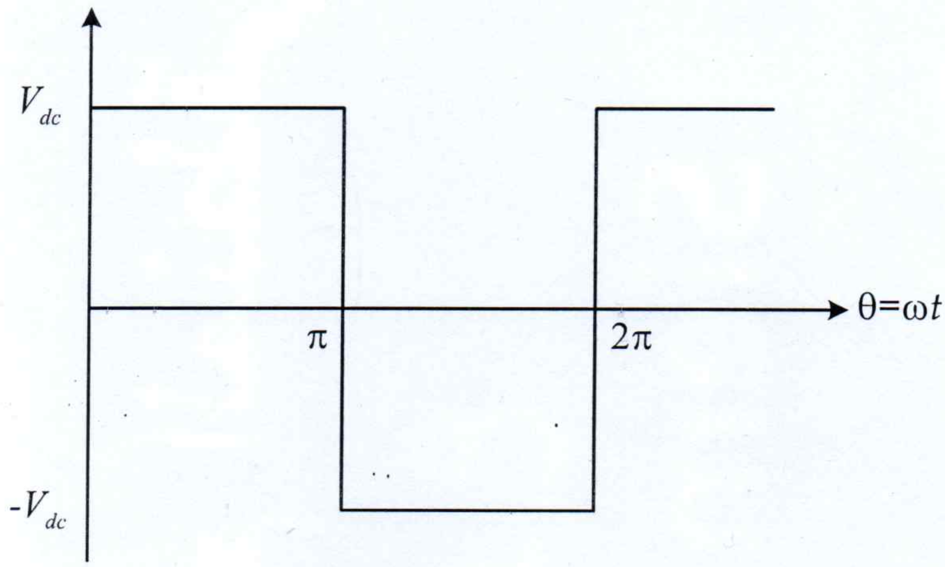
$$= \frac{5\cancel{\pi}}{\cancel{\pi}} - \frac{\pi^{\cancel{2}}}{2\cancel{\pi}}$$

$$= 5 - \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi}(5-x)\cos nx\,dx + \cancel{\frac{1}{\pi}\int_{\pi}^{2\pi} 0\,dx}$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} 5\cos nx\,dx - \int_0^{\pi} x\cos nx\,dx\right]$$

Harmonics of square-wave (1)



$$a_o = \frac{1}{\pi}\left[\int_0^{\pi} V_{dc}\,d\theta + \int_{\pi}^{2\pi} -V_{dc}\,d\theta\right] = 0$$

$$a_n = \frac{V_{dc}}{\pi}\left[\int_0^{\pi} \cos(n\theta)\,d\theta - \int_{\pi}^{2\pi} \cos(n\theta)\,d\theta\right] = 0$$

$$b_n = \frac{V_{dc}}{\pi}\left[\int_0^{\pi} \sin(n\theta)\,d\theta - \int_{\pi}^{2\pi} \sin(n\theta)\,d\theta\right]$$

Harmonics of square wave (2)

Solving,

$$b_n = \frac{V_{dc}}{n\pi}\left[-\cos(n\theta)\Big|_0^{\pi} + \cos(n\theta)\Big|_{\pi}^{2\pi}\right]$$

$$= \frac{V_{dc}}{n\pi}\left[(\cos 0 - \cos n\pi) + (\cos 2n\pi - \cos n\pi)\right]$$

$$= \frac{V_{dc}}{n\pi}\left[(1 - \cos n\pi) + (1 - \cos n\pi)\right]$$

$$= \frac{2V_{dc}}{n\pi}\left[(1 - \cos n\pi)\right]$$

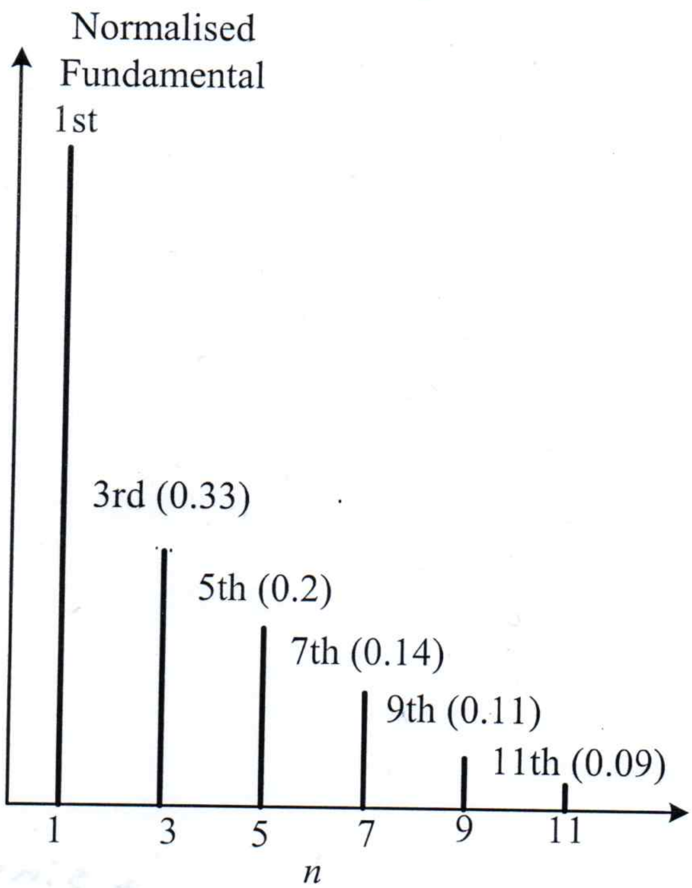
when n is even, $\cos n\pi = 1$

$b_{n=}0$

when n is odd, $\cos n\pi = -1$

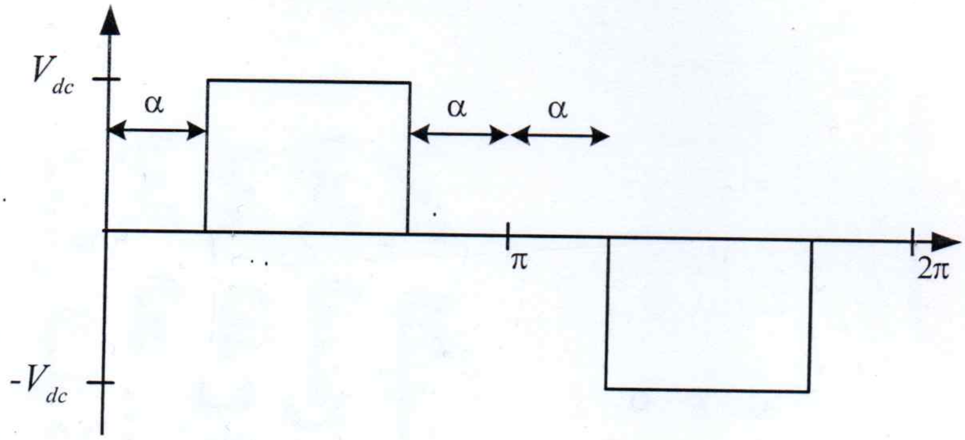
$b_{n=}\dfrac{4V_{dc}}{n\pi}$

Spectra of square wave



- Spectra (harmonics) characteristics:
  - Harmonic decreases as *n* increases. It decreases with a factor of $(1/n)$.
  - Even harmonics are absent
  - Nearest harmonics is the 3rd. If fundamental is 50Hz, then nearest harmonic is 150Hz.
  - Due to the small separation between the fundamental an harmonics, output low-pass filter design can be quite difficult.

Quasi-square wave (QSW)



Note that $a_n = 0$.

Due to half - wave symmetry,

$$b_n = 2\left[\frac{1}{\pi}\int_{\alpha}^{\pi-\alpha} V_{dc}\sin(n\theta)\,d\theta\right] = \frac{2V_{dc}}{n\pi}\left[-\cos n\theta\Big|_{\alpha}^{\pi-\alpha}\right]$$

$$= \frac{2V_{dc}}{n\pi}\left[\cos(n\alpha) - \cos n(\pi-\alpha)\right]$$

Expanding,

$$\cos n(\pi-\alpha) = \cos(n\pi - n\alpha)$$
$$= \cos n\pi\cos n\alpha + \sin n\pi\sin n\alpha$$
$$= \cos n\pi\cos n\alpha$$

$\sin n(\pi - \alpha) = \sin(n\pi - n\alpha)$

$= \underbrace{\sin n\pi}_{=0}\cos n\alpha - \underbrace{\cos n\pi}_{=-1}\sin n\alpha$

$= \sin n\alpha$

Harmonics control

$$\Rightarrow b_n = \frac{2V_{dc}}{n\pi}\left[\cos(n\alpha) - \cos n\pi \cos n\alpha\right]$$

$$= \frac{2V_{dc}}{n\pi}\cos(n\alpha)[1 - \cos n\pi]$$

If n is even, $\Rightarrow b_n = 0,$

If n is odd, $\Rightarrow b_n = \frac{4V_{dc}}{n\pi}\cos(n\alpha)$

In particular, amplitude of the fundamental is :

$$b_1 = \frac{4V_{dc}}{\pi}\cos(\alpha)$$

The fundamental, b_1, is controlled by varying α

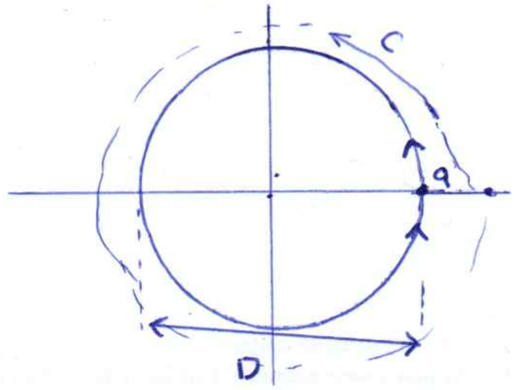
Harmonics can also be controlled by adjusting α, For example if $\alpha = 30^o$, then $b_3 = 0$, or the third harmonic is eliminated from the waveform. In general, harmonic n will be eliminated if :

$$\alpha = \frac{90^o}{n}$$

Radian & Degree

Pi (π) / is the ratio of a circle's circumference to its diameter. It is approximately = 3.14.

$$Pi = \frac{C}{D} = 3.14$$



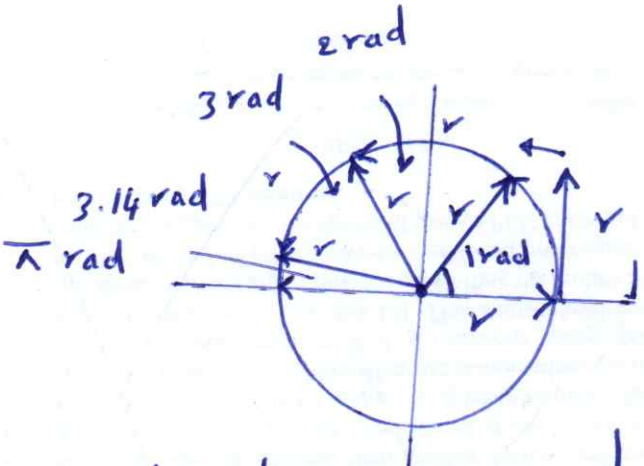
$aa = 3.14D$

Radian / is the unit for measuring angles.

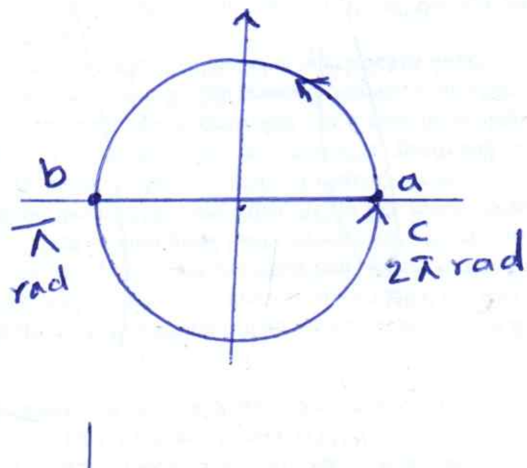
π rad = 180°

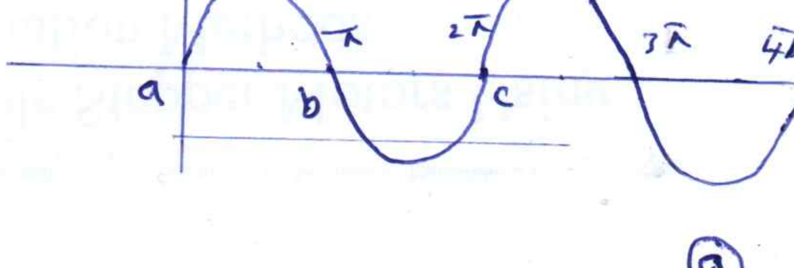
2π rad = 360°

1 rad = 57.32°



1 cycle = 2π rad





Regular frequency (f) & Angular frequency (ω)

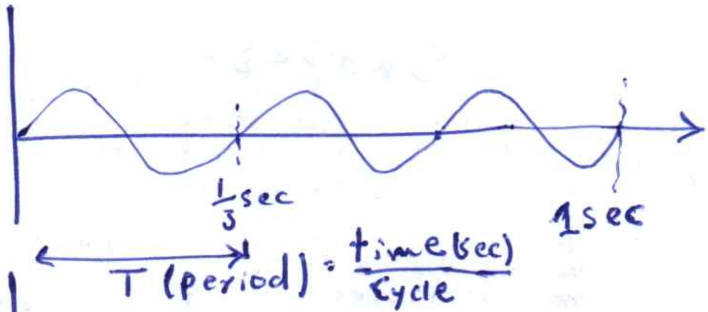
$\boxed{\text{cycle/sec or Hz}}$ $\qquad$ $\boxed{\text{rad/sec}}$

- Regular frequency (f) is measured in cycles per second (Hz)

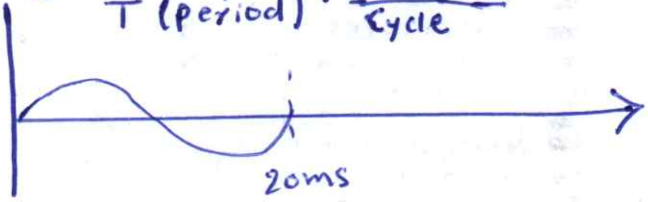
$$f(Hz) = \cancel{\frac{\text{no. of cycle}}{1\ \text{sec}}} = \frac{1\ \text{cycle}}{T(\text{time for 1 cycle})}$$

$\cancel{f = \frac{3}{1\ \text{sec}} = 3Hz}$ →

$$f = \frac{1}{T} = \frac{1}{\frac{1}{3}s} = 3Hz$$



$$f = \frac{1}{20ms} = 50Hz \rightarrow$$



- Angular frequency (ω) (rad/sec)

$$\omega = \frac{1\ \text{angular displacement}}{T\ (\text{time in sec})\ \text{for one cycle}}\ \text{rad/sec}$$

$$\omega = \frac{2\pi}{\frac{1}{3}\text{sec}} = 6\pi \Longrightarrow \boxed{\omega = 2\pi f = \frac{2\pi}{T}}$$

$$\omega = \frac{2\pi}{20\ ms} = 100\pi$$

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