Laplace Transformation:

- The laplace transform is one of mathematical tools for solving the (linear ordinary and partial) differential equations by converting them into algebraic equations which are ofen easier to solve.

Differintional equation $\xrightarrow{\text{by}}$ Laplace transform $\rightarrow$ Algebraic equation $\rightarrow$ Algebraic techniques $\rightarrow$ Inverse Laplace $\rightarrow$ obtain the solution in terms of original variable

- Laplace transform is an integral transform that converts a function of a real variable (usually t in time domain) to a function of a complex variable s (freqnency domain or s-domain or s-plane).

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

Laplace

$f(t) \longrightarrow F(s)$

$1 \longrightarrow \frac{1}{s}$ see page (5)

$c \longrightarrow \frac{c}{s}$ $\quad c = \text{constant}$

$t^n \longrightarrow \frac{n!}{s^{n+1}}$

$e^{at} \longrightarrow \frac{1}{s-a}$ $\qquad e^{-at} \longrightarrow \frac{1}{s+a}$

$\sin wt \rightarrow \frac{w}{s^2+w^2}$ $\qquad \cos wt \rightarrow \frac{s}{s^2+w^2}$

$t^n f(t) \longrightarrow (-1)^n \frac{d^n}{ds^n}\left[F(s)\right]$

$\mathcal{L} \int_0^t f(t)\,dt \longrightarrow \frac{1}{s} \mathcal{L} f(t)$

H.W : prove that :-

$\mathcal{L} \cos wt = \frac{s}{s^2+w^2}$

ex)

1- $\mathcal{L}\,3 = \frac{3}{s}$

2- $\mathcal{L}\,\frac{5}{6} = \frac{5/6}{s}$

3- $\mathcal{L}\,t^2 = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$

4- $\mathcal{L}\,3t^5 = \frac{3 \times 5!}{s^6}$

5- $\mathcal{L}\,e^{3t} = \frac{1}{s-3}$

6- $\mathcal{L}\,2e^{-4t} = \frac{2}{s+4}$

7- $\mathcal{L}\,4\sin 5t = \frac{4 * 5}{s^2 + 25}$

8. $\mathcal{L}\,2\cos 3t = \frac{2s}{s^2+9}$

9. $\mathcal{L}\,[t - e^{3t}] = \frac{1}{s^2} - \frac{1}{s-3}$

10- $\mathcal{L}\left[2e^{-4t} + 3t^3 - \sin 5t\right] = \frac{2}{s+4} + \frac{3 * 3!}{s^4} - \frac{5}{s^2+25}$

11- $\mathcal{L}\,e^{3t}\sin 7t$

$\mathcal{L}\,\sin 7t = \frac{7}{s^2+49}$

$\mathcal{L}\,e^{3t}\sin 7t = \frac{7}{(s-3)^2 + 49}$

12- $\mathcal{L}\, e^{-2t}\cos 8t$

$$\mathcal{L}\cos 8t = \frac{s}{s^2+64}$$

$$\mathcal{L}\, e^{-2t}\cos 8t = \frac{(s+2)}{(s+2)^2+64}$$

13- $\mathcal{L}\, t\, e^{2t}$

$$\mathcal{L}\, t = \frac{1}{s^2} \qquad t^n f(t) \to (-1)^n \frac{d^n}{ds^n} F(s)$$

$$\mathcal{L}\, t\, e^{2t} = \frac{1}{(s-2)^2}$$

14- $\mathcal{L}\, t\, e^{2t}$

$$\mathcal{L}\, e^{2t} = \frac{1}{s-2}$$

$$\mathcal{L}\, t\, e^{2t} = -\frac{d}{ds}\left[\frac{1}{(s-2)}\right]$$

$$= -\frac{(s-2)*0 - 1(1)}{(s-2)^2} = -\frac{-1}{(s-2)^2}$$

15- $\mathcal{L}[4t^2 - 3\cos 2t + 5e^{-t}$

$$= \frac{8}{s^3} + \frac{3s}{s^2+4} + \frac{5}{s+1}$$

ex) $\mathcal{L}\, C \qquad f(t) = c$

$$F(s) = \mathcal{L}\, c = \int_0^\infty c e^{-st}\, dt = c \int_0^\infty e^{-st}\, dt$$

$$\int e^{ax}\, dx = \frac{e^{ax}}{a} + c$$

$$F(s) = -\frac{c}{s}\, e^{-st} \Big|_0^\infty$$

$$= -\frac{c}{s} \left(\underbrace{e^{-\infty}}_{0} - \underbrace{e^{-0}}_{1} \right)$$

$$= \frac{-c}{s}(-1) = \frac{9}{s}$$

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ex) $\mathcal{L}(t-3)^3$

$$= \mathcal{L}(t-3)^2(t-3)$$

$$= \mathcal{L}(t^2 - 6t + 9)(t-3)$$

$$= \mathcal{L}\, t^3 - 3t^2 - 6t^2 + 18t + 9t - 27$$

$$= \mathcal{L}\, t^3 - 9t^2 + 27t - 27$$

$=$

* **Shifting theorem :-** $\boxed{\mathcal{L}[e^{at}\cdot f(t)] = F(s-a)}$

(First Shifting property)

ex) $\mathcal{L}\, e^{-2t}\, t^3$

$$\mathcal{L}\, t^3 = \frac{3!}{s^4} = \frac{6}{s^4}$$

s+2 ← s بدل

$$\mathcal{L}\, e^{-2t} = \frac{1}{s+2}$$

$$\mathcal{L}\, e^{-2t}\, t^3 = \frac{6}{(s+2)^4}$$

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ex) $\mathcal{L}\, e^{3t} \sin 4t$

$$\mathcal{L}\, e^{3t} = \frac{1}{s-3}$$

s-3 ← s بدل

$$\mathcal{L} \sin 4t = \frac{4}{s^2+4^2} = \frac{4}{s^2+16}$$

$$\mathcal{L}\, e^{3t} \sin 4t = \frac{4}{(s-3)^2+16}$$

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ex) $\mathcal{L}\, e^{2t} \cos 3t$

$$\mathcal{L}\, e^{2t} = \frac{1}{s-2}$$

s-2 ← s بدل

$$\mathcal{L} \cos 3t = \frac{s}{s^2+9}$$

$$\mathcal{L}\, e^{2t} \cos 3t = \frac{s-2}{(s-2)^2+9}$$

ex) $\mathcal{L}\, e^{3t} \cdot t^2$

$\mathcal{L}\, t^2 = \frac{2}{s^3}$ $\mathcal{L}\, e^{3t} = \frac{1}{s-3}$

$\mathcal{L}\, e^{3t} \cdot t^2 = \frac{2}{(s-3)^3}$

* Multiplication by t^n

If $\mathcal{L}\, f(t) = F(s)$ then

$$\boxed{\mathcal{L}\left[t^n \cdot f(t)\right] = (-1)^n \cdot \frac{d^n}{ds^n} F(s)}$$

ex) $\mathcal{L}\, t \cdot e^{2t}$

$$\mathcal{L}\, t = \frac{1}{s^2} \qquad \mathcal{L}\, e^{2t} = \frac{1}{s-2}$$

$$\mathcal{L}\, t \cdot e^{2t} = \frac{1}{(s-2)^2}$$

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or

$$\mathcal{L}\left[t \cdot e^{2t}\right] = -1^{1} \cdot \frac{d}{ds}\left(\frac{1}{s-2}\right) = -\frac{-1}{(s-2)^2} = \frac{1}{(s-2)^2}$$

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ex) $\mathcal{L}\, t^2 \cos at \qquad \mathcal{L}\, \cos at = \dfrac{s}{s^2+a^2}$

$$\mathcal{L}\, t^2 \cdot \cos at = -1^2 \cdot \frac{d^2}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= \frac{d}{ds}\left[\frac{s^2+a^2 \quad - \quad 2s}{(s^2+a^2)^2}\right]$$

$$=$$

ex) $\mathcal{L}\, t \sin t$

$$= -1 \frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= -\left[\frac{(s^2+1)*0 - 2s}{(s^2+1)^2}\right] = -\left[\frac{-2s}{(s^2+1)^2}\right]$$

$$= \frac{2s}{(s^2+1)^2}$$

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ex) $\mathcal{L}\, t^2 \sin 3t$

$$= (-1)^2 \frac{d^2}{ds^2}\left[\frac{3}{s^2+9}\right]$$

$$= \frac{d}{ds}\left[\frac{0 - 3 * 2s}{(s^2+9)^2}\right] = \frac{d}{ds}\left[\frac{-6s}{s^2+9}\right]$$

$$= \frac{(s^2+9)^2 * -6 - (-6s) * 2(s^2+9).2s}{(s^2+9)^4}$$

$$= \frac{-6(s^2+9)^2 + 24s^2(s^2+9)}{(s^2+9)^4}$$

$$= \frac{\cancel{(s^2+9)}\left[-6(s^2+9)+24s^2\right.}{(s^2+9)^{\cancel{4}\,3}}$$

$$= \frac{-6(s^2+9)+24s^2}{(s^2+9)^3}$$

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ex) $\mathcal{L}\, t\, e^{2t} \cos 4t$

$$= -1 \frac{d}{ds}\left[\mathcal{L}\, e^{2t} \cos 4t\right]$$

$$= -1 \frac{d}{ds}\left[\frac{s}{s^2+16}\Big|_{s=s-2}\right]$$

$$= -1 \frac{d}{ds}\left[\frac{s-2}{(s-2)^2+16}\right]$$

$$= -1\left[\frac{((s-2)^2+16)*(1) - (s-2)*(2(s-2)*1)}{((s-2)^2+16)^2}\right]$$

$$= -\left[\frac{(s-2)^2+16-2(s-2)^2}{((s-2)^2+16)^2}\right]$$

$$= -\left[\frac{-(s-2)^2+16}{((s-2)^2+16)^2}\right.$$

$$= \frac{(s-2)^2-16}{((s-2)^2+16)^2}$$

* Inverse Laplace Transformation:

$$\mathcal{L}^{-1} F(s) = f(t)$$

- $\mathcal{L}^{-1} \frac{a}{s} = a$
- $\mathcal{L}^{-1} \frac{s}{s^2+a^2} = \cos at$
- $\mathcal{L}^{-1} \frac{n}{s^{n+1}} = t^n$
- $\mathcal{L}^{-1} \frac{a}{s^2+a^2} = \sin at$
- $\mathcal{L}^{-1} \frac{1}{s \mp a} = e^{\pm at}$

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ex) $\mathcal{L}^{-1} \frac{4}{s^5} = 4 \mathcal{L}^{-1} \frac{1}{s^5} * \frac{4!}{4!}$

$$= \frac{4}{4!} \mathcal{L}^{-1} \frac{4!}{s^5} = \frac{\cancel{4}^{1}}{\cancel{24}_{6}} t^4$$

$$= \frac{t^4}{6}$$

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ex) $\mathcal{L}^{-1} \frac{1}{s^6}$

$$= \mathcal{L}^{-1} \frac{5!}{s^6} \cdot \frac{1}{5!} = \frac{1}{5!} t^5$$

$$= \frac{t^5}{120}$$

ex)

$$\mathcal{L}^{-1} \frac{4-3s}{s^2+9}$$

$$= \mathcal{L}^{-1}\left[\frac{4}{s^2+9} - \frac{3s}{s^2+9}\right]$$

$$= 4\,\mathcal{L}^{-1} \frac{1}{s^2+9} - 3\,\mathcal{L}^{-1} \frac{s}{s^2+9}$$

$$= 4\,\mathcal{L}^{-1} \frac{3}{s^2+3^2} \cdot \frac{1}{3} - 3\,\mathcal{L}^{-1} \frac{s}{s^2+3^2}$$

$$= \frac{4}{3} \sin 3t - 3 \cos 3t$$

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ex)

$$\mathcal{L}^{-1} \frac{(s+3)}{(s+3)^2+16} = \mathcal{L}^{-1} \frac{(s+3)}{(s+3)^2+(4)^2}$$

$$= \cos 4t \cdot e^{-3t}$$

ex) $\mathcal{L}^{-1} \dfrac{1}{s^2 + 3s}$

$$\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$

$$\frac{1}{s(s+3)} = \frac{A(s+3) + Bs}{s(s+3)}$$

$$As + 3A + Bs = 1 \quad = \quad s(A+B) + 3A = 1$$

$$A + B = 0$$

$$3A = 1 \longrightarrow A = \frac{1}{3}$$

$$\frac{1}{3} + B = 0 \longrightarrow B = \frac{-1}{3}$$

$$\frac{1}{3}\mathcal{L}^{-1}\frac{1}{s} - \frac{1}{3}\mathcal{L}^{-1}\frac{1}{s+3}$$

$$= \frac{1}{3} - \frac{1}{3}e^{-3t}$$

$$= \frac{1}{3} - \frac{e^{-3t}}{3}$$

Laplace Inverse :

ex) $\mathcal{L}^{-1} \dfrac{s+1}{s^2-3s+2}$

sol) $\dfrac{s+1}{s^2-3s+2} = \dfrac{s+1}{(s-1)(s-2)} = \dfrac{A}{s-1} + \dfrac{B}{s-2}$

$\begin{bmatrix} s^2-3s+2 \\ s^2-2s-s+2 \end{bmatrix}$ $= \dfrac{A(s-2)+B(s-1)}{(s-1)(s-2)}$

$s+1 = A(s-2) + B(s-1) = As - 2A + Bs - B$

$s+1 = s(A+B) - 2A + B$

$1 = A + B$
$1 = -2A + B$ $\Longrightarrow$ $A = -2$, $B = 3$

$\mathcal{L}^{-1} \dfrac{s+1}{(s-1)(s-2)} = \mathcal{L}^{-1} \dfrac{-2}{s-1} + \mathcal{L}^{-1} \dfrac{3}{s-2}$

$= -2e^{t} + 3e^{2t}$

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ex) $\mathcal{L}^{-1} \dfrac{s^2+1}{(s+2)^3}$

$\dfrac{s^2+1}{(s+2)^3} = \dfrac{A}{(s+2)} + \dfrac{B}{(s+2)^2} + \dfrac{C}{(s+2)^3}$

$s^2+1 = A(s+2)^2 + B(s+2) + C$

$s^2+1 = As^2 + 4sA + 4 + Bs + 2B + C$

$s^2+1 = As^2 + s(4A+B) + 4 + 2B + C$

$s^2 = As^2 \longrightarrow \boxed{A = 1}$

$4A + B = 0 \longrightarrow \boxed{B = -4}$

$4 + 2B + C = 1$

$4 + 2(-4) + C = 1$

$\boxed{C = 5}$

$$\mathcal{L}^{-1}\frac{s^2+1}{(s+2)^3} = \mathcal{L}^{-1}\frac{1}{s+2} + \mathcal{L}^{-1}\frac{-4}{(s+2)^2}$$ 15

$$+ \mathcal{L}^{-1}\frac{5}{(s+2)^3}$$

$$= e^{-2t} - 4\,e^{-2t}\cdot t + \frac{5}{2}\,e^{-2t}\cdot t^2$$

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ex) $\mathcal{L}^{-1}\dfrac{s+3}{s^2-2s-3}$

$$\frac{s+3}{(s-3)(s+1)} = \frac{A}{(s-3)} + \frac{B}{(s+1)}$$

$$\frac{s+3}{(s-3)(s+1)} = A(s+1) + B(s-3)$$

$$s+3 = As + A + Bs - 3B$$

$$A + B = 1 \qquad \longrightarrow \qquad A = \frac{3}{2} \qquad B = \frac{-1}{2}$$

$$A - 3B = 3$$

$$\mathcal{L}^{-1}\frac{\frac{3}{2}}{s-3} - \frac{1}{2}\mathcal{L}^{-1}\frac{1}{s+1}$$

$$= \frac{3}{2}e^{3t} - \frac{1}{2}e^{-t}$$
